PROJECTIVE PROPERTIES OF CONICS

by

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INTRODUCTION

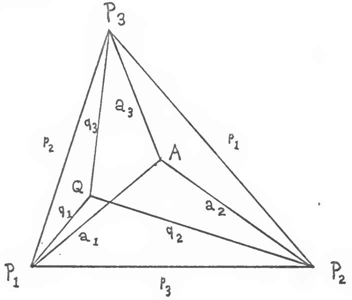
This report contains a development of some of the projective properties of the conics. The particular properties investigated are those pertaining first to conjugate points and lines, second to poles and polars, and third to self polar triangles. Conjugate points have the following properties: Two points, the first of which lies on the conic, are conjugate if and only if the second point lies on the tangent at the first, and two points neither of which lies on the conic are conjugate if and only if they separate harmonically the points of intersection of their line with the conic. Conjugate lines have properties corresponding to those of conjugate points.

Poles and polars, which are developed from conjugate points and lines, exhibit the following property: The pole of the line joining two points is the point of intersection of the polars of the two points, and the polar of the point of intersection of two lines is the line joining the poles of the two lines. The concept of poles and polars leads to the notion of self polar triangles.

A triangle is self polar relative to a conic if and only if each pair of sides are conjugate lines relative to the conic, and each pair of vertices are conjugate points relative to the conic. The report concludes with the result that there are only two types of conics in the projective plane, those with real traces and those without real traces.

PROJECTIVE COORDINATES

For a system of projective coordinates in a plane consider the four points P_1, P_2, P_3, Q, no three of which are collinear. Let p_1, p_2, p_3 be the sides of the triangle $p_1p_2p_3$ and q_1, q_2, q_3 be the lines joining the point Q to the vertices of the triangle. Let A be an arbitrary point and let a_1, a_2, a_3 be the lines which join the point A to P_1, P_2, P_3 respectively.



Consider the cross ratios:

(1) $\alpha_1 = (p_2p_3,\ q_1a_1)$, $\qquad \alpha_2 = (p_3p_1,\ q_2a_2)$,

$$\alpha_3 = (p_1p_2,\ q_3a_3),$$

where $(L_1L_2,\ L_3L_4)$ means the ratio that L_3 divides L_1 and L_2 divided by the ratio that L_4 divides L_1 and L_2 with L_1, L_2, L_3, L_4 distinct concurrent lines.[1]

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[1] William C. Graustein, Introduction to Higher Geometry, p. 72.

The numbers α_1, α_2, α_3 could be used as coordinates for the point A except that if A were on one of the sides of the triangle, then at least one of the three cross ratios would be undefined. This difficulty is overcome by using for the coordinates of A any three numbers (x_1, x_2, x_3) such that

$$(2) \quad \alpha_1 = x_3/x_2, \qquad \alpha_2 = x_1/x_3, \qquad \alpha_3 = x_2/x_1.$$

Then there are no exceptional points which have to be defined in some special manner. The coordinates (x_1, x_2, x_3) are called homogeneous projective coordinates of the point A.

Consider the general homogeneous equation of second degree in projective coordinates x_1, x_2, x_3:

$$(3) \qquad \sum_{i,j=1}^{3} p_{ij}x_ix_j = 0, \qquad p_{ij} = p_{ji}\,.$$

If the coefficients p_{ij} of equation (3) are all zero, then the locus of points whose coordinates satisfy the equation is the set of all points on the plane. If the coefficients are real and not all zero, then the locus of points whose coordinates satisfy equation (3) is called a <u>point conic.</u>

Suppose $\sum_{i,j=1}^{3} p_{ij}x_ix_j$ can be factored into two linear factors as follows:

$$(4) \quad \sum_{i,j=1}^{3} p_{ij}x_ix_j = \left(\sum_{i=1}^{3} a_ix_i\right)\left(\sum_{i=1}^{3} b_ix_i\right).$$

Then the locus of equation (3) is the locus of

$$(5) \qquad \sum_{i=1}^{3} a_i x_i = 0,$$

together with that of

$$(6) \qquad \sum_{i=1}^{3} b_i x_i = 0.$$

Equations (5) and (6) represent two straight lines which may be distinct and parallel, coincident, or intersect in one point. In any of these cases the point conic is called degenerate. If this is not the case, that is, if equation (3) cannot be factored into linear factors, then the point conic is called nondegenerate. A necessary and sufficient condition that a point conic be nondegenerate is that $|p_{ij}| \neq 0$.[2] Since this report deals with nondegenerate conics, the added condition $|p_{ij}| \neq 0$ is assumed hereafter and a nondegenerate point conic is called simply a point conic.

Let a and b, represented by (a_1, a_2, a_3) and (b_1, b_2, b_3), be two distinct points on a straight line L. An arbitrary point on the straight line L is given by

$$(7) \qquad x = \alpha a + \beta b, \quad \alpha, \beta \text{ arbitrary constants not both zero.}$$

This arbitrary point lies on the conic given by equation (3) if and only if

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[2]Graustein, op. cit., p. 190.

$$(8) \qquad \sum_{i,j=1}^{3} p_{ij}(\alpha a_i + \beta b_i)(\alpha a_j + \beta b_j) = 0.$$

Since $p_{ij} = p_{ji}$ equation (8) implies

$$(9) \qquad \alpha^2 \sum_{i,j=1}^{3} p_{ij} a_i a_j + 2\alpha\beta \sum_{i,j=1}^{3} p_{ij} a_i b_j$$

$$+ \beta^2 \sum_{i,j=1}^{3} p_{ij} b_i b_j = 0,$$

which is a homogeneous quadratic equation in α, β with constant coefficients. Equation (9) determines the coordinates of the two points $\alpha_1 a + \beta_1 b$, and $\alpha_2 a + \beta_2 b$ which are common to the line and the conic provided the coefficients are not all zero. If the coefficients are all zero, then the lines (7) and the conic coincide.

<u>THEOREM:</u> A straight line always intersects a nondegenerate point conic in two points, distinct or coincident.[3]

A tangent line to a conic is a line which intersects the conic in two coincident points. Since the two points coincide, $\alpha_1 a + \beta_1 b$ and $\alpha_2 a + \beta_2 b$ are the same.

A secant line to a conic is a line which intersects the conic in two distinct points. Then the two points $\alpha_1 a + \beta_1 b$ and $\alpha_2 a + \beta_2 b$ are not the same.

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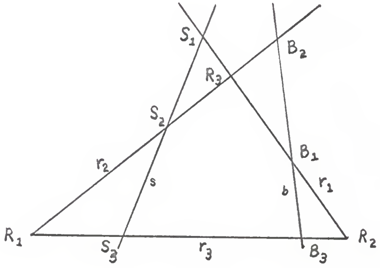
[3] Graustein, <u>op</u>. <u>cit</u>., p. 192.

If the equation of a conic is equation (3) and a is a point on the conic, then the equation of the tangent to the conic at the point a is

$$(10) \qquad \sum_{i,j=1}^{3} p_{ij}a_i x_j = 0, \qquad |p_{ij}| \neq 0.$$[4]

For a system of projective line coordinates in a plane consider the four real lines r_1, r_2, r_3, s, no three of which are concurrent. Let r_1, r_2, r_3 be the sides of a triangle and R_1, R_2, R_3 the vertices opposite r_1, r_2, r_3 respectively. Let S_1, S_2, S_3 be the pointsof intersection of s with r_1, r_2, r_3. Choose an arbitrary line b and let B_1, B_2, B_3 be the points of intersection of b with r_1, r_2, r_3.



Form the cross ratios

$$(11) \qquad \beta_1 = (R_2R_3, S_1B_1), \qquad \beta_2 = (R_3R_1, S_2B_2),$$

$$\beta_3 = (R_1R_2, S_3B_3),$$

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[4]Graustein, <u>op. cit.</u>, p. 193.

where (P_1P_2, P_3P_4) means the ratio that P_3 divides the line segment P_1P_2 divided by the ratio that P_4 divides the line segment P_1P_2, with P_1, P_2, P_3, P_4 distinct collinear points.[5] Represent these cross ratios in terms of v_1, v_2, v_3 as follows:

$$(12) \qquad \beta_1 = v_3/v_2, \quad \beta_2 = v_1/v_3, \quad \beta_3 = v_2/v_1 .$$

The coordinates (v_1, v_2, v_3) are called homogeneous line coordinates of the line b.

Consider the equation of the second degree with v_1, v_2, v_3 projective line coordinates

$$(13) \qquad \sum_{i,j=1}^{3} q_{ij}v_iv_j = 0, \qquad q_{ij} = q_{ji} .$$

If the coefficients are all real and not all zero, then the totality of lines whose coordinates satisfy equation (13) is called a <u>line conic</u>.

A line conic may consist of two points which are distinct or coincident. If the line conic is two points, distinct or coincident, then the line conic is called <u>degenerate</u>; otherwise it is called nondegenerate. A line of a line conic is a singular line if every point determined by it and a line of the conic belongs to the line conic. A <u>nondegenerate</u> line conic, then, has no singular line. A necessary and sufficient condition that a line conic be degenerate is that $|q_{ij}| = 0$. Since this report deals with nondegenerate conics, hereafter it is assumed that $|q_{ij}| \neq 0$. A

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[5] Graustein, <u>op</u>. <u>cit</u>., p. 74.

nondegenerate line conic is called simply a line conic.

The dual of a point conic and its tangent lines is the line conic and its contact points. If equation (13) is the equation of a line conic and r is a line on the conic, then the equation of the contact point of the line conic on the line r is

$$(14) \qquad \sum_{i,j=1}^{3} q_{ij} r_i v_j = 0, \qquad |q_{ij}| \neq 0.$$ [6]

There is a correspondence between point conics and line conics which is indicated in the following theorem.

THEOREM: The set of all tangents to a point conic is a line conic, and the set of all contact points of a line conic is a point conic.[7]

One can speak of a conic then in terms of either line coordinates or point coordinates. If the equation of a conic in point coordinates is

$$(15) \qquad \sum_{i,j=1}^{3} p_{ij} x_i x_j = 0, \qquad p_{ij} = p_{ji}, \qquad |p_{ij}| \neq 0,$$

then the corresponding equation in line coordinates is

$$(16) \qquad \sum_{i,j=1}^{3} P_{ij} v_i v_j = 0, \qquad P_{ij} = P_{ji}, \qquad |P_{ij}| \neq 0,$$

with P_{ij} the cofactor of p_{ij} in $|p_{ij}|$.

If the equation of the conic in line coordinates is

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[6] Graustein, op. cit., p. 197.

[7] C. W. O'Hara and D. R. Ward, Projective Geometry, p. 116.

(17) $$\sum_{i,j=1}^{3} q_{ij}v_iv_j = 0, \qquad q_{ij} = q_{ji}, \qquad |q_{ij}| \neq 0,$$

then the corresponding equation in point coordinates is

(18) $$\sum_{i,j=1}^{3} Q_{ij}x_ix_j = 0, \qquad Q_{ij} = Q_{ji}, \qquad |Q_{ij}| \neq 0,$$

with Q_{ij} the cofactor of q_{ij} in $|q_{ij}|$.

The point conics with their tangent lines and the line conics with their contact points are considered to be identical, and the point conic and line conic is hereafter called a conic.[8]

CONJUGATE POINTS AND LINES

There are three major classes of properties of conics: Projective, affine, and metric. Only the first of the three is considered in this report. Conjugate points and lines are the first properties investigated, since they serve as groundwork for other properties.

A conic is represented by either of the two equations

(19a) $$\sum_{i,j=1}^{3} p_{ij}x_ix_j = 0, \qquad p_{ij} = p_{ji}, \qquad |p_{ij}| \neq 0,$$

or

(19b) $$\sum_{i,j=1}^{3} q_{ij}v_iv_j = 0, \qquad q_{ij} = q_{ji}, \qquad |q_{ij}| \neq 0,$$

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[8] Graustein, op. cit., p. 199.

where x_i, x_j are point coordinates and v_i, v_j line coordinates. These two equations will be occasionally referred to as equation (19), and the next two as equation (20). If equation (19a) is given first then $q_{ij} = P_{ij}$, the cofactor of p_{ij}. If equation (19b) is given first, then $p_{ij} = Q_{ij}$, the cofactor of q_{ij}.

If the coordinates of two points a and b satisfy either

(20a) $$\sum_{i,j=1}^{3} p_{ij} a_i b_j = 0,$$

or

(20b) $$\sum_{i,j=1}^{3} p_{ij} b_i a_j = 0,$$

then a and b are called <u>conjugate points</u> with respect to the conic represented by equation (19a). Two conjugate points are related to the conic in a manner stated in the following two theorems.

<u>THEOREM 1:</u> Two points, the first of which lies on the conic, are conjugate if and only if the second point lies on the tangent at the first.

Proof: Let the two points be a and b. If a and b are conjugate, then equation (20) is true. If a is assumed to be the point on the conic, then b is on the tangent at a since equation (10) is the equation of the tangent to the conic at the point a.

If b lies on the tangent at a then equation (20a) is true and the points a and b are conjugate by definition.

<u>THEOREM 2:</u> Two points, neither of which lie on the conic, are conjugate if and only if they separate harmonically the intersection of

their line with the conic.

Proof: Let the two points neither of which is on the conic be a and b. The two points determine a line and the points where that line crosses the conic are x_1 and x_2 whose coordinates are given by

(21) $$x_1 = a + \alpha_1 b, \qquad x_2 = a + \alpha_2 b,$$

with α_1, and α_2 the roots of the quadratic equation

(22) $$\sum_{i,j=1}^{3} p_{ij} a_i a_j + 2\alpha \sum_{i,j=1}^{3} p_{ij} a_i b_j + \alpha^2 \sum_{i,j=1}^{3} p_{ij} b_i b_j = 0.$$

If a and b are separated harmonically by the points x_1 and x_2, then by definition $\alpha_1 + \alpha_2 = 0$. Since the sum of the roots of the quadratic equation is zero the coefficient of α in the middle term of equation (22) is zero, and that is the condition that a and b be conjugate.

If a and b are conjugate, then the middle term of equation (22) is zero. Hence the sum of the roots of equation (22) is zero and a and b are separated harmonically.

Two lines r and s which satisfy

(23a) $$\sum_{i,j=1}^{3} q_{ij} r_i s_j = 0,$$

or

(23b) $$\sum_{i,j=1}^{3} q_{ij} s_i r_j = 0,$$

are called conjugate lines with respect to the conic represented by equation (19b). Theorems 1 and 2 have two corresponding theorems which are now given in terms of conjugate lines.

THEOREM 3: Two lines, the first of which is a tangent to a conic, are conjugate if and only if the second passes through the point of contact of the first with the conic.

Proof: Let the two lines be r and s, and let r be the tangent to the conic.

If r and s are conjugate then equations (23a) and (23b) are true. The line s then passes through the point of contact of r with the conic because equation (14) is the equation of the point of contact of the line r.

If s passes through the point of contact of r with the conic, then equation (23a) is true and r and s are conjugate lines by definition.

THEOREM 4: Two lines, neither of which is a tangent to the conic, are conjugate if and only if they separate harmonically the tangents from their point of intersection.

Proof: Suppose that neither r nor s is a tangent to the conic. Let v_1 and v_2 be the tangents to the conic from the point of intersection of r and s. The lines v_1 and v_2 are given by

(24) $$v_1 = r + \beta_1 s, \qquad v_2 = r + \beta_2 s,$$

with β_1 and β_2 roots of the quadratic equation

(25) $$\sum_{i,j=1}^{3} q_{ij}r_ir_j + 2\beta \sum_{i,j=1}^{3} q_{ij}r_is_j + \beta^2 \sum_{i,j=1}^{3} q_{ij}s_is_j = 0.$$

If r and s separate harmonically the tangents from the point of intersection, then $\beta_1 + \beta_2 = 0$. Since the sum of the roots of equation (25) is zero, the coefficient of the middle term is zero and that is the condition that r and s be conjugate.

If r and s are conjugate the middle term of equation (25) is zero, hence the sum of the roots of the equation is zero and r and s are separated harmonically.

POLES AND POLARS

For the discussion of poles and polars, the conic is considered to be given by equation (19). Poles and polars are developed from two viewpoints: First, they are developed from conjugate points, and second they are developed from conjugate lines.

Given a point a, then a point x is conjugate to a with respect to the conic in equation (19) if and only if

(26) $$\sum_{i,j=1}^{3} p_{ij}a_ix_j = 0.$$

Consider the locus of all points x which are conjugate to a given point a with respect to the conic in equation (19). The locus is a line v. The line v is called the <u>polar</u> of the point a with respect to the conic and the polar of a is given by equation (26).

THEOREM 5: If a point is on the conic, then the polar of the point is the tangent at the point. If a point is not on the conic, then the polar of the point is the secant joining the points of contact of the tangents from the point to the conic.

Proof: Let the point a be on the conic, then the polar of the point a is given by equation (26). Equation (26) also represents the tangent to the conic at the point a. Thus the polar of a point a on the conic is the tangent at the point a.

If the point a is not on the conic, then there are two points which are the points of contact of the tangents from a. The two points of contact define a line which is a secant of the conic. Each point of contact is conjugate to the point a. Hence the secant determined by the point of contact is the polar of the point a with respect to the conic.

If a point a has the line v as its polar, then a is called the pole of v.

THEOREM 6: If a line is tangent to a conic, then the pole of the line is the point of tangency to the conic. If a line is a secant of a conic, then the pole of the line is the point of intersection of the tangents to the points in which the secant meets the conic.

Proof: If the line v is tangent to the conic, then v is represented by equation (10) which is the same as equation (26). Hence the line v is the polar of point a and the point a is the point of tangency.

If the line is a secant of the conic then the points of intersection with the conic are conjugate to the point of intersection of the tangents from these points. All points conjugate to a given point are on the polar of the point which in this case is the secant. Hence the given point is the pole of the secant.

Poles and polars are now developed from the viewpoint of conjugate lines. Given a fixed line v, a line y is conjugate to v if and only if

$$\sum_{i,j=1}^{3} q_{ij} v_i y_j = 0 . \tag{27}$$

The totality of lines y which are conjugate to a given line v with respect to the conic in equation (19) pass through a point a. The point a is called the pole of the line v with respect to the conic, and the pole a is given by equation (27).

THEOREM 7: If a line is tangent to a conic, then the pole of the line is the point of tangency to the conic. If a line is a secant of a conic, then the pole of the line is the point of intersection of the tangents to the points in which the secant meets the conic.

Proof: If a line v is a tangent to the conic, then the point of contact is represented by equation (14) which is the same as equation (27), and equation (27) gives the pole of the line.

If a line is a secant, the tangents to the conic at the point of intersection of the secant are conjugate to the secant: The two tangents meet at a point which is the pole of the secant.

If a point a is given, there is a line v which has as its pole the point a. The line v is called the polar of a with respect to the conic.

THEOREM 8: If a point is on the conic, then the polar of the point is the tangent at the point. If a point is not on the conic, then the polar of the point is the secant joining the points of contact of the tangents from the point to the conic.

Proof: If the point a is on the conic, then the polar of a is represented by equation (26) which is the same as equation (10), hence the polar of the point is the tangent at the point.

If the point is not on the conic, then there are two lines through the point and tangent to the conic. Their points of contact determine a secant line intersecting the conic at the points of tangency. The point not on the conic is the pole of this secant, hence the secant is the polar of the point.

It is to be noted that Theorem 5 and Theorem 8 are identical and Theorem 6 and Theorem 7 are identical even though they are developed differently. Hence one can speak of poles and polars without regard to whether the development is based on conjugate points or conjugate lines.

THEOREM 9: A point is on a conic if and only if it lies on its polar. A line is tangent to a conic if and only if it contains its pole.

Proof: If a point is on a conic, by Theorem 5 its polar is the tangent at the point. Hence the point lies on its polar. If a point lies on its polar, by definition the polar is tangent to the conic at the point and the point is on the conic.

If a line is tangent to a conic, by Theorem 6 its pole is the point of tangency and hence the line contains its pole. If the line contains its pole, by definition the line is tangent to the conic. Hence the point of tangency is the pole of the line.

THEOREM 10: If a first point lies on the polar of a second point, then the second point lies on the polar of the first point. If a first point passes through the pole of a second line, the second line passes through the pole of the first line.

Proof: If one point lies on the polar of a second point, then by the definition of conjugate points, the first point is conjugate to the second point. By the same definition of conjugate points, the second point lies on the polar of the first point since the polar contains all conjugate points.

If one fixed line passes through the pole of a second line, the fixed line is conjugate to the second line by the definition of conjugate lines. Then the second line is conjugate to the first line and from the definition of a pole, the second line passes through the pole of the first line.

THEOREM 11: The pole of the line joining two points is the point of intersection of the polars of the two points. The polar of the point of intersection of two lines is the line joining the poles of the two lines.

Proof: Let p be the pole of the line l which is determined by the two points a_1 and a_2. The line l is the polar of p and since the two points a_1 and a_2 lie on l, the pole p lies on the polar of a_1 and a_2. Since p lies on both of the polars, it is the point of intersection of the polars of a_1 and a_2.

Let l be the polar of the point a which is the point of intersection of two lines k_1 and k_2. The point a is the pole of l and since the two lines k_1 and k_2 intersect at a, the polar of a lies on the pole of k_1 and k_2. Since l meets the pole of both k_1 and k_2, l must be the line joining the two poles.

The following theorem is an extension of Theorem 11.

THEOREM 12: The polars of any number of points which lie on a line all go through a point which is the pole of the line. The poles of any number of lines which all pass through a point, lie on a line which is the polar of the point.

Proof: Consider the polars of two distinct points P_1, P_2 on a line L. By Theorem 11 the polars of P_1, P_2 pass through a point P which is the pole of the line. Consider then one of the first points, say P_1, and any other point P' on the line L. By Theorem 11 the polars of P_1 and P' pass through a point Q which is the pole of the line L. The pole of a line is unique so P = Q.

Consider two lines which pass through a point. By Theorem 11 the poles of the two lines lie on a line which is the polar of the point of intersection of the two lines. Consider one of the two original lines and any other line which passes through the point of intersection of the first two lines. By Theorem 11 the poles of the two lines lie on a line which is the polar of the point of intersection of the two lines. But the polar of a point is unique so the polar in each case is the same.

SELF POLAR TRIANGLES

If p is a point of the plane, and q is any point on its polar with respect to a given conic, and r is the intersection of the polars of p and q, then the triangle pqr is such that each vertex is the pole of the opposite side and each side is the polar of the opposite vertex. Such a triangle is called self polar relative to the given conic.

THEOREM 13: A triangle is self polar relative to a conic if and only if each pair of sides are conjugate lines relative to the conic, and each pair of vertices are conjugate points relative to the conic.

Proof: If a triangle is self polar relative to a conic, then each side is the polar of the vertex opposite it. Each vertex is conjugate to the other two vertices since they lie on its polar. Hence any two vertices are conjugate points. The pole of each side is the vertex opposite it. Thus the intersection of two sides is the pole of the third side. Hence the polar is conjugate to each of the other sides, and each pair of sides are conjugate lines.

If a pair of vertices in a triangle are conjugate relative to a conic then the remaining vertex is the pole of the side opposite it. Thus each vertex is the pole of the side opposite it. If a pair of sides in a triangle are conjugate relative to a conic, then the remaining side is the polar of the opposite vertex. So each side is the polar of the vertex opposite it. Hence by definition the triangle is self polar.

Self polar triangles are used as triangles of reference for the reduction of the equation of a conic to an equation in standard form. Let the equation of a conic be

$$(28) \qquad \sum_{i,j=1}^{3} p_{ij}x_i x_j = 0, \quad p_{ij} = p_{ji}, \qquad |p_{ij}| \neq 0.$$

Consider a triangle pqr that is a self polar triangle relative to the conic represented by equation (28). Introduce new projective coordinates (x_1', x_2', x_3') with pqr the triangle of reference. In the new coordinate system equation (28) becomes

$$(29) \qquad \sum_{i,j=1}^{3} p'_{ij}x'_i x'_j, \quad p'_{ij} = p'_{ji}, \qquad |p'_{ij}| \neq 0.$$

The pairs of vertices of triangle pqr are conjugate points. Also the vertices of triangle pqr have in the new coordinate system the coordinates (1, 0, 0), (0, 1, 0), (0, 0, 1).[9] Two distinct points a and b are conjugate with respect to the conic represented by equation (29) if and only if

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[9]Graustein, op. cit., p. 157.

(30) $$\sum_{i,j=1}^{3} p'_{ij}a_i b_j = 0.$$

Since (1, 0, 0) and (0, 1, 0) are conjugate points with respect to the conic represented by equation (29), $p'_{12} = 0$, and by definition $p'_{12} = p'_{21}$. In a similar manner $p'_{23} = p'_{32} = 0$, and $p'_{13} = p'_{31} = 0$. So equation (29) reduces to

(31) $$p'_{11}x'^2_1 + p'_{22}x'^2_2 + p'_{33}x'^2_3 = 0,$$

$$p'_{11}p'_{22}p'_{33} \neq 0.$$

Equation (31) is now reduced to an equation in standard form by another change of coordinates. There are two cases to be considered. If all of the coefficients of equation (31) are of the same sign they are assumed to be all positive. If they were all negative, multiplying equation (31) by -1 would make them all positive. So the linear transformation

(32) $$\sigma x_1'' = \sqrt{p'_{11}}\ x_1', \qquad \sigma x_2'' = \sqrt{p'_{22}}\ x_2',$$

$$\sigma x_3'' = \sqrt{p'_{33}}\ x_3',$$

reduces equation (31) to

(33) $$x_1''^2 + x_2''^2 + x_3''^2 = 0.$$

If not all of the coefficients are of the same sign, it is assumed two are positive and one is negative. A preassigned coefficient is made negative by either multiplying equation (31) by -1 or renaming the vertices of triangle pqr, or both. So assume p'_{11} and p'_{22} are positive and p'_{33} is negative. Then the linear transformation

$$(34) \qquad \sigma x_1'' = \sqrt{p'_{11}}\ x_1', \quad \sigma x_2'' = \sqrt{p'_{22}}\ x_2', \quad \sigma x_3'' = \sqrt{p'_{33}}\ x_3',$$

reduces equation (31) to

$$(35) \qquad x_1''^2 + x_2''^2 - x_3''^2 = 0.$$

Equation (33) represents an equation in point coordinates of a conic which has no real points, and equation (35) represents an equation in point coordinates of a conic which has real points. This results in the following theorem:

THEOREM 14: The equation of a conic is reducible by a change of projective coordinates to

$$(36a) \qquad x_1^2 + x_2^2 + x_3^2 = 0,$$

if the conic has not a real trace, or

(36b) $$x_1^2 + x_2^2 - x_3^2 = 0,$$

if the conic has a real trace.

The equations in line coordinates of a conic which correspond respectively to equations (36a) and (36b) are

(37a) $$v_1^2 + v_2^2 + v_3^2 = 0,$$

and

(37b) $$v_1^2 + v_2^2 - v_3^2 = 0.$$

Equation (36a) represents a conic which is determined by a change of coordinates from equation (28). The aforementioned properties of the conic are preserved in the transformation so all conics without a real trace can be represented as one type of conic. Also all conics with a real trace can be represented as one type of a conic. So there are only two types of conics in the projective plane, those with real traces and those without real traces.

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PROJECTIVE PROPERTIES OF CONICS

by

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AN ABSTRACT OF A MASTER'S REPORT

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This report contains a development of some of the projective properties of conics. As background material for the projective properties, the definition of a point conic and a line conic is given in terms of projective coordinates. The particular properties which are discussed are those pertaining to conjugate points and lines, poles and polars, and self polar triangles.

Conjugate points have the following properties: Two points, the first of which lies on the conic, are conjugate if and only if the second point lies on the tangent at the first, and two points neither of which lies on the conic are conjugate if and only if they separate harmonically the points of intersection of their line with the conic. Conjugate lines have the following properties which correspond to those of conjugate points: Two lines, the first of which is a tangent to a conic, are conjugate if and only if the second passes through the point of contact of the first with the conic, and two lines, neither of which is a tangent to the conic, are conjugate if and only if they separate harmonically the tangents from their point of intersection.

Poles and polars are developed from conjugate points and lines. Poles and polars exhibit the following properties: A point is on a conic if and only if it lies on its polar, and a line is tangent to a conic if and only if it contains its pole. In addition, the following is true: The pole of the line joining two points is the point of intersection of the polars of the two points, and the polar of the point of intersection of two lines is the line joining the poles of the two lines. The concept of poles and polars leads to the development of self polar triangles.

A triangle is self polar relative to a conic if and only if each pair of sides are conjugate lines relative to the conic, and each pair of vertices are conjugate points relative to the conic. The report concludes with a self polar triangle being used as a triangle of reference to reduce the equation of a conic to standard form. This yields the result that there are only two types of conics in the projective plane, those with real traces and those without real traces.