THE LEBESGUE INTEGRAL

Cassette Notes I
(Containing M331 TV and Radio Programmes 1–4)
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Prepared by the Course Team
A covering note

M431 Cassette Notes I and II have been compiled directly from the Radio and Television Broadcast Notes of M331 to accompany the cassettes of those same programmes, which we are sending you.

The resources for producing M431 have been minimal and could not be stretched to remake any of the radio or television programmes. However, a review of this material revealed that it contains much that will be extremely useful to you, vindicating the decision to send you the material unchanged from its original form.

This means that you will have to translate the M331 unit numbers into their M431 counterparts. You will have to overlook references to Units 15 and 16 of M331 as they do not form part of this course, and also references to OU courses that no longer exist. Our decision also means that there is no audio-visual material for Unit 2 of this course, which is completely new. The full list of M431 units will be found in the Guide to the Course.

Finally, we must point out a mathematical error in the radio programme covering Unit 12, Hilbert Space. The definition of a Schauder basis given there is incomplete. Because the definition of linear independence involves only finite sets of vectors at a time, it is possible to have too many vectors in a set purported to be a Schauder basis. This will result in the nonuniqueness of Fourier expansion coefficients. The correct definition includes the additional statement of uniqueness of the expansion coefficients. With this proviso, the programme is correct as it stands.

An example of this phenomenon is given in Unit 12 of this course, where a certain extra function $e_0$ is added to a legitimate Schauder basis $T = \{e_1, e_2, \ldots\}$.

The resulting set $T' = \{e_0, e_1, e_2, \ldots\}$ remains linearly independent, but the additional function now has two different expansions in terms of $T'$:

$$e_0 = 0 \cdot e_0 + \sum_{n=1}^{\infty} a_n e_n,$$

where the $a_n$ are the Fourier coefficients of $e_0$ in its expansion with respect to $T$; and

$$e_0 = 1 \cdot e_0 + \sum_{n=1}^{\infty} 0 \cdot e_n.$$

Hence $T'$ is not a Schauder basis using our (correct) definition, but satisfies the (incorrect) definition of the programme.

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THE LEBESGUE INTEGRAL

Introduction to the Programme
The intention is to show why we need to extend the Riemann Integral and to indicate the method adopted in M331 to achieve this extension. It is not essential to have read any of the Correspondence Texts before viewing this programme, but you will find it helpful to have studied the Guide to the Course.

Presentation of the Programme

Earlier Integrals
Allan Solomon introduces the programme by reminding you that M331 is about a new sort of integration called Lebesgue integration. Ian Dey looks at the ways in which ideas about integration have developed out of the concept of area. He recalls how in M100 we defined the integral of a continuous function by approximating the area under the graph of the function by sums of rectangular areas and then taking limits. This is essentially the approach introduced by Cauchy around 1820 and developed by Riemann about 1870; it is the integration used in M231, and is usually called Riemann Integration. In M331 we shall consider an extension of this integral which will take into account limits of sequences of functions.
The Lebesgue Integral

To see the sort of properties we would like the Lebesgue integral to possess, Allan looks at those properties of the Riemann integral which we would like to retain. He concludes that we would like the set of Lebesgue integrable functions (the Lebesgue domain) to be a vector space and would like the Lebesgue integral to be a linear operator which agrees with the Riemann integral on the set of Riemann integrable functions (the Riemann domain).

What sort of functions are we likely to want to integrate that are not already Riemann integrable? Allan considers the following function, $f: \mathbb{R} \rightarrow \mathbb{R}$:

$$
\begin{align*}
f : x & \mapsto \\
& \begin{cases}
1 & \text{if } x \text{ is a rational number belonging to } [0,1] \\
0 & \text{otherwise}
\end{cases}
\end{align*}
$$

and proves that it is not Riemann integrable. You may think that this is not the sort of function that anyone would want to integrate but functions like this appear quite often in applications as limits of sequences of Riemann integrable functions and it would be useful if they could be integrated too. In this particular case $f$ is the limit of the sequence $\{\phi_i\}$ where $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$
\begin{align*}
\phi_i : x & \mapsto \\
& \begin{cases}
1 & \text{if } x \text{ belongs to } [0,1] \text{ and can be expressed as a fraction } \frac{n}{m} \text{ with } m \leq i. \\
0 & \text{otherwise}
\end{cases}
\end{align*}
$$
and each $\phi_i$ is Riemann integrable with $\int \phi_i = 0$. We would therefore like $f$ to be integrable with $\int f = \lim \{\int \phi_i\} = 0$.

Ian Dey examines what can go wrong by looking at the diagram

For the example above we are unable to complete the diagram across the top because the function $f = \lim \{\phi_n\}$ is not Riemann integrable and so we cannot assert that $\int \lim \{\phi_n\} = \lim \{\int \phi_n\}$. To remedy this we would like the Lebesgue definition of the integral to be such that the integration operator commutes with the limit operator.

**Defining the Lebesgue Integral**

To summarize, the Lebesgue integral should have all the useful properties of the Riemann integral and should also have the important property that the integration operator and the limit operator commute. Allan explains that this last property can in fact help us to define the Lebesgue integral in terms of the integrals of Step functions as follows:
If $\{\phi_n\}$ is an increasing sequence of step functions such that the sequence of real numbers $\{f_{\phi_n}\}$ is bounded and

$$f = \lim \{\phi_n\}$$

then define

$$\int f = \lim \{\int \phi_n\}.$$

We have used step functions in this definition because it is easy to see how they can be integrated. By choosing increasing sequences of step functions $\{\phi_n\}$ it follows that $\{\int \phi_n\}$ is increasing and this, along with $\{\int \phi_n\}$ bounded, guarantees that $\lim \{\int \phi_n\}$ exists.

Ian takes a closer look at the definition and points out that some increasing sequences of step functions $\{\phi_n\}$ with bounded integrals do not necessarily converge at every point of the real line and although it may be tempting to exclude these sequences from the definition such an approach turns out to be too restrictive. What we do is change our concept of convergence: at those points $x$ at which $\{\phi_n\}$ converges we take $f(x) = \lim (\phi_n(x))$ but at those points at which $\{\phi_n\}$ diverges we allow $f(x)$ to take any value we like. We then say that $\{\phi_n\}$ converges to $f$ almost everywhere. With this definition of limit there may be infinitely many limit functions for a given sequence $\{\phi_n\}$ but for each limit $f$ we define $\int f = \lim \{\int \phi_n\}$. This approach is not as arbitrary as it may appear because the set of values at which $\{\phi_n\}$ diverges is a very small set. It is small in the sense that every point in the set can be covered by a sequence of intervals the total length of which is arbitrarily small. Such sets are called Null Sets.
Taking this new concept of convergence Allan modifies his definition of integral in the following way. The integral of a function \( f \), if it has one, is found by producing an increasing sequence of step functions \( \{ \phi_n \} \) with bounded integrals which converges \textit{almost everywhere} to \( f \). We then define \( \int f = \lim \{ \int \phi_n \} \).

The only remaining problem with this definition is that it may not be consistent, that is there may be several increasing sequences of step functions with bounded integrals which converge almost everywhere to \( f \) and each one may give a different answer for the integral. The fact that the definition is consistent is the content of an important theorem in Unit 3.

The set of functions that can be integrated using the above definition is denoted by \( L^{\text{inc}} \). It is not the whole of the Lebesgue domain but only the 3rd stage in a four stage process. Ian Dey has a brief look at each of these stages. They are:

1. Defining the integral of the characteristic function of a bounded interval.

2. Defining the integral of step functions.

3. Defining the integral of functions in \( L^{\text{inc}} \) as above.

4. Defining the integral of functions in the vector space \( L^1 \) spanned by \( L^{\text{inc}} \).

The programme ends with an animated sequence summarizing these four stages.
Broadcast Self-Assessment Question (BSAQ)

Which of the following options describes one difference between the domain $R$ of the Riemann integral, and the domain $L^1$ of the Lebesgue integral?

Options

1. $R$ is a vector space, but $L^1$ is not.

2. $R$ contains only continuous functions, while $L^1$ contains some discontinuous ones.

3. $R$ does not contain the limits of certain sequences of integrable functions, which it might reasonably be expected to contain, while $L^1$ contains such limit functions.
Solutions to BSAQ

Option 1 is false: it is true that \( R \) is a vector space and \( \text{Linc} \) is not, but the domain \( L^1 \) of the Lebesgue integral that we eventually construct is a vector space.

Option 2 is false: both \( R \) and \( L^1 \) contain discontinuous functions (though \( L^1 \) does contain some discontinuous functions which are not contained in \( R \)).

Option 3 is true: one advantage of \( L^1 \) is that it contains the limits of sequences of (Lebesgue) integrable functions (subject to certain conditions which we shall investigate later) while, in general, \( R \) does not.
RADIO PROGRAMME 1

COMPLETENESS

Introduction to the Programme

This is an introductory programme which is mainly concerned with an important property which we wish our new definition of the integral to possess: the completeness property of the Lebesgue integral.

It is not essential to have read any of the correspondence texts before listening to the programme. However, you should read the notes in Section A before the broadcast. The notes in Section B are to be followed during the broadcast. In Section C, we give a detailed summary of the programme to be used for revision after the broadcast: it is not intended that you should try to follow this summary during the broadcast.

A Notes to be read before the broadcast

In this programme we refer to the concepts of distance function and metric space.

Although these concepts were not defined explicitly in Spivak, you are in fact familiar with one example of a metric space, namely the set of real numbers together with the usual distance function \( d \) defined by

\[ d(x,y) = |x-y|, \quad x, y \in \mathbb{R}. \]

(If you have studied M201 or M202 you will have met other examples in Unit M201 16, Euclidean Spaces I, and Unit M202 7, Metric Space Axioms.)
The metric space of real numbers with the above distance function has the following (completeness) property:

Every increasing sequence of real numbers \( \{s_n\} \) that is bounded above converges to a real number, \( \ell \) say: that is, for any \( \varepsilon > 0 \), there exists a positive integer \( N \) such that

\[ |s_n - \ell| < \varepsilon \quad \text{for all } n > N. \]

In the broadcast we generalize these ideas by introducing the concept of \textbf{metric completeness} (this is defined in the Appendix to Section B) and relate them to the concept of the Lebesgue integral.

(You may wish to read the Appendix to Section B before the broadcast, where we define the concepts underlined above.)

\[ \text{B Notes to be followed during the broadcast} \]

1. **Completeness Axiom (Spivak's property P13; see Spivak : page 113)**

Any non-empty subset of the real numbers that is bounded above has a least upper bound.

2. **Completeness Axiom (Equivalent to P13)**

An increasing sequence of real numbers that is bounded above converges to a limit.
f : R → R, f continuous for a ≤ x ≤ b

3
f(b) > 0

4
Upper bound

5
Least upper bound

If f is continuous on a closed bounded interval [a, b] such that f(a) < 0 and f(b) > 0, then f must vanish somewhere in [a, b].

A function which is continuous on a closed interval [a, b] is bounded above in the interval.

A function continuous on a closed interval [a, b] attains its least upper bound.

f : Q → Q, f continuous for 0 ≤ x ≤ 2

3a

4a

5a

Counter-example:

f(x) = x^2 - 2

Counter-example:

f(x) = 1/(x^2 - 2)

Counter-example:

f(x) = (6x - x^3)^2

6 Complements for Lebesgue integrable functions

An increasing sequence \( \{f_n\} \) of Lebesgue integrable functions whose sequences of integrals \( \{\int f_n\} \) is bounded above has a limit function (a.e.) which is Lebesgue integrable (and \( \lim \{\int f_n\} = \int \lim \{f_n\} \)).
"Distance" between $f$ and $g = \sqrt{\int (f-g)^2}\). This distance function gives rise to a Euclidean Space (i.e. a metric space in which there is a notion of orthogonality).

Appendix to Section B

Distance Function

A distance function, or metric, on a set $X$ is a real function $d: X \times X \rightarrow \mathbb{R}$ such that for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x)$;

(ii) $d(x, y) = 0$ if, and only if, $x = y$;

(iii) $d(x, y) + d(y, z) \geq d(x, z)$ (Triangle Inequality).

Metric Space

A metric space is a pair $(X, d)$ consisting of a non-empty set $X$ and a distance function $d$.

Metric Completeness

(a) Limit Point: The sequence $\{x_n\}$ of points of a metric space $(X, d)$ has a limit point $x \in X$, if, for any $\varepsilon > 0$, there exists a positive integer $N$ such that, for all $n > N$, $d(x, x_n) < \varepsilon$.

(b) Cauchy Sequence: The sequence $\{x_n\}$ of points of a metric space $(X, d)$ is a Cauchy Sequence if, for any $\varepsilon > 0$,
there exists a positive integer N such that for all m > N, n > N, \( d(x_m, x_n) < \varepsilon \).

(c) Metric Completeness: The Metric Space \((X, d)\) is Complete if every Cauchy sequence \(\{x_n\}\) of \(X\) has a limit point \(x \in X\).

C Summary of Programme

In this programme Allan Solomon explains the need for a new approach to integration theory. He points out that Lebesgue integration is more powerful than the integration you studied in M100 or M231. It is more powerful in the sense that it enlarges the set of functions that can be integrated, in such a way that this larger space of functions has a property of "completeness" which is analogous to the completeness of the real numbers.

Completeness in the Reals

Before discussing the importance of completeness in the context of integration, Allan describes the completeness properties of the real numbers. In M231 the completeness property was presented in the form of an axiom which Spivak calls Axiom P.13. In M331 we shall use a different formulation of this axiom, as given in Note 2. Both formulations are equivalent but the latter is more appropriate for the construction of the Lebesgue integral.
To demonstrate the importance of completeness for the reals, Allan recollects the "three hard theorems" of Spivak; they are illustrated in diagrams 3, 4 and 5 in Section B. Many important results in analysis and calculus depend on the properties stated in these three theorems. The validity of these theorems depends on the completeness property of the reals. In the rational number system, Q, which does not satisfy the completeness axiom, none of these theorems is true. This is illustrated by the counter-examples in diagrams 3a, 4a and 5a where each function is continuous (in Q) but has domain and codomain Q. Allan goes on to contrast these theoretical considerations of completeness with computational considerations. Using the calculation of the area of a circle as an example he points out that in numerical calculations we have to work with rational numbers. The fact that the rational numbers are not complete does not prevent us from doing numerical work.

A similar state of affairs is found in Integration theory; most familiar functions can be integrated using Riemann integration but the completeness property of Lebesgue integration gives it a theoretical advantage over Riemann integration analogous to that which the real number system possesses over the rationals.

Completeness of the Lebesgue Integral

Returning to Lebesgue integration, Allan formulates a completeness theorem for the Lebesgue integral (Note 6 above). It is similar to the completeness axiom for the reals given in Note 2, but it is a theorem and not an axiom.
As an illustration of the importance of the completeness theorem, Allan considers the approximation of functions by sequences of simpler functions such as polynomial functions or step functions. It is useful to know that certain sequences of such functions have integrable limit functions. This property is not true for Riemann integrable functions in general.

Another example of the importance of completeness arises in connection with vector spaces of functions. Such spaces can sometimes be turned into "metric spaces" by defining a distance function or "metric" on them. There are many ways of doing this, but Allan recalls one particular way discussed in Unit M201 16, Euclidean Spaces I:

The "distance" between two functions \( f \) and \( g \) is defined to be \( [\int (f - g)^2]^\frac{1}{2} \).

It turns out that, if the Lebesgue integral is used in this definition, then the resulting space will possess an extremely powerful completeness property called "metric completeness". Roughly speaking, metric completeness means that, given any sequence of functions in the space that get "closer and closer together" (in a way which is defined precisely later in this course), this sequence has a limit function which is also in the space. (Notice that we need the metric to give us the idea of closeness.) A precise formulation of this axiom is
given in the appendix to Section B, where the concept has been
generalized to any metric space. In fact the distance function
defined in Note 7 not only gives a metric space with completeness but
also introduces the idea of orthogonality; that is, it gives a
complete Euclidean Space or Hilbert space. These spaces were discussed
briefly in Unit M201 20, Euclidean Spaces II; they are extremely use-
ful and important in Modern Theoretical Physics.

D Broadcast Self-Assessment Question (BSAQ)
Which option most effectively summarizes this radio programme?

Options

1. The Lebesgue integral is useful because it enables us to evaluate
   integrals more easily.

2. The Lebesgue integral is a powerful theoretical construct because
   the space of Lebesgue integrable functions has completeness
   properties analogous to those of the real numbers.

3. The Lebesgue integral is more powerful than the Riemann integral
   because it can be used to construct distance functions.
Solution to BSAQ

Option 1 is false: in practice, there is no computational advantage afforded to the Lebesgue integral over the Riemann integral.

Option 2 is true: this is the great theoretical advantage of the Lebesgue integral.

Option 3 is false: we can also construct distance functions using the Riemann integral. However, in general, the resulting metric spaces will not be metric complete.
Introduction to the Programme

The intention is to look carefully at all the ingredients which make up the definition of the integral on $L^1$. The conditions on the defining sequences of step functions are examined, and the resulting convergence behaviour discussed.

Presentation of the Programme

The four stages

Ian Dey introduces the programme by reminding you of the four stages involved in the construction of the Lebesgue Integral and then looks briefly at the construction of the integral in the first two stages. In the rest of the programme we concentrate on the 3rd stage, $L^1$, and look carefully at the definition of the integral in $L^1$.

Definition of the Integral in $L^1$

Roger Duke recalls that, roughly speaking, the integral of a function in $L^1$ is defined in terms of sequences of step functions which converge to the function concerned. The trouble is that in order to make the definition work it is necessary to place certain restrictions on the type of sequences of step functions we use; it is also necessary to modify our ideas of convergence.
Ian states the definition which we arrived at in programme 1:

\[ f \text{ belongs to } L^\text{inc} \text{ if it is the limit almost everywhere of a sequence of step functions } \{ \phi_n \}. \]

The conditions that \( \{ \phi_n \} \) must satisfy are that it must be increasing and that the corresponding sequence of integrals \( \{ f\phi_n \} \) must be bounded and so convergent. For every such \( f \) we define \( f = \lim_n \{ f\phi_n \} \).

Before this definition is examined in more detail Roger looks at two specific examples which illustrate how the definition works in practice.

First of all he looks at the function

\[
f : x \mapsto \begin{cases} 
1 & \text{if } x \in \mathbb{Q}[0,1] \\
0 & \text{otherwise}
\end{cases}
\]

and observes that it is the limit of the increasing sequence of step functions \( \{ \phi_n \} \) where

\[
\phi_n : x \mapsto \begin{cases} 
1 & \text{if } x \text{ belongs to } [0,1] \text{ and can be expressed as a fraction } \frac{i}{m} \\
0 & \text{otherwise}
\end{cases}
\]
Since $\int \phi_n = 0$ for all $n$, the definition tells us that $\int f = \lim \{ \int \phi_n \} = 0$
(this result was anticipated in programme 1).

As a second example he considers the characteristic function $\chi_{[0,1]}$.

From the definition of the integral of a characteristic function of a bounded interval we know that $\int \chi_{[0,1]} = 1$ but it is worth checking that the same answer is obtained by using the definition above. To do this we note that $\chi_{[0,1]}$ is the limit of the increasing sequence of step functions

$$
\phi_n : x \rightarrow \begin{cases} 
1 - \frac{1}{n} & \text{if } x \in [0,1] \\
0 & \text{otherwise.}
\end{cases}
$$

Furthermore $\{ \int \phi_n \}$ is bounded, and so $\int \chi_{[0,1]} = \lim \{ \int \phi_n \} = \lim \{ \frac{1}{n} \} = 1$, as expected.
Conditions placed on the Step Functions

Returning to the definition, Ian looks in more detail at the conditions placed on the step functions \( \{ \phi_n \} \). There are two conditions, \( \{ \phi_n \} \) increasing and \( \{ f \phi_n \} \) bounded, and together they guarantee that \( \lim \{ f \phi_n \} \) exists, but in fact they do more than this as we can see by trying to vary the conditions.

If \( \{ \phi_n \} \) is an increasing sequence

If \( \{ f \phi_n \} \) converges

If \( \{ \phi_n \}^+ \) and \( \{ f \phi_n \} \) bdd.

and \( f = \lim \phi_n \) a.e.

define \( f_\phi = \lim \{ f \phi_n \} \)

First of all we might think that the two conditions could be replaced by the single condition "\( \{ f \phi_n \} \) converges". To see why this is not sufficient Roger looks at the sequence of step functions

\[
\phi_n(x) = \begin{cases} 
  n & \text{if } x \in (0, \frac{1}{n}) \\
  0 & \text{otherwise.}
\end{cases}
\]

Now this satisfies the condition "\( \{ f \phi_n \} \) converges"; in fact \( f \phi_n = 1 \) for all \( n \) and so \( \lim \{ f \phi_n \} = 1 \). The problem is that \( \{ \phi_n \} \) converges to the zero function which implies that \( \lim f \phi_n = 0 = 0 \), and so we cannot assert that \( \lim \{ f \phi_n \} = \lim \{ f \phi_n \} \). The thing that has gone wrong is that \( \{ \phi_n \} \) is not an increasing sequence.
Another possibility is that we just use the simple condition that "\( \{\phi_n\} \) is an increasing sequence". Roger examines this by looking at the increasing sequence of step functions

\[
\phi_n : x \mapsto \begin{cases} 
1 & \text{if } x \in [-n, n] \\
0 & \text{otherwise.}
\end{cases}
\]

This converges to the function \( \chi_R : x \mapsto 1 \), but \( \int \phi_n = 2n \) and so \( \{\int \phi_n\} \) does not converge. In this case we are unable to assert that \( \lim \{\phi_n\} = \lim \{\int \phi_n\} \) because \( \lim \{\int \phi_n\} \) does not exist. The thing that has gone wrong this time is that \( \{\int \phi_n\} \) is unbounded.

It appears, then, that both conditions are required for the definition to work.

**Convergence of the Step Functions**

Ian now goes on to consider the other unusual feature of the definition; it concerns the convergence of the sequence of step functions. The definition attempts to define \( \mathcal{L}^{\text{inc}} \) to be the set of all limit functions of increasing sequences of step functions with bounded integrals. However, not all sequences with these properties have limit functions.

To see the sort of thing that can go wrong, Roger considers the sequence

\[
\phi_n : x \mapsto \begin{cases} 
n & \text{if } x = 1, 2, 3, \ldots n \\
\chi_{(0,1)}(x) & \text{otherwise.}
\end{cases}
\]
This is an increasing sequence of step functions with bounded integrals but it does not converge at every point of the real line; in fact it diverges at every positive integer. The interesting point here is that the positive integers form a null set. In Unit 3 it is shown that any sequence of step functions with the properties described above diverge on at most a null set, that is, they converge almost everywhere.

Ian points out that this lack of convergence on null sets is not disastrous because intuitively we would not expect the integral of a function to be affected by its behaviour on a null set. To show that this is plausible he demonstrates that we would expect the integral of a characteristic function of a null set to be zero. It appears that convergence almost everywhere to a limit function is in fact the most natural thing to use in the definition of the integral. Furthermore the existence of such a limit is guaranteed by the conditions that have already been placed on the step functions.

By looking again at sequence \( \{ \phi_n(x) \} \) above Roger investigates how convergence almost everywhere affects the definition. If \( f \) is a function which takes arbitrary values on the positive integers and is equal to \( \lim \{ \phi_n(x) \} \) elsewhere, then \( \{ \phi_n \} \) converges almost everywhere to \( f \) and so \( \int f = \lim \{ \int \phi_n \} = 1 \). If \( g \) is any other function which differs from \( f \) only on the positive integers then \( \{ \phi_n \} \) also converges almost everywhere to \( g \) and again \( \int g = \lim \{ \int \phi_n \} = 1 \).
a limit almost everywhere of \( \{ \phi_n \} \)

In fact, we do not have to restrict ourselves to the points at which \( \{ \phi_n \} \) diverges; we can change the values of \( f \) on any null set and the resulting function will still be the limit almost everywhere of \( \{ \phi_n \} \) and will have an integral equal to \( \lim \{ \int \phi_n \} = 1 \). In general an increasing sequence of step functions \( \{ \phi_n \} \), with bounded integrals, will have a set of functions which are all limits almost everywhere of \( \{ \phi_n \} \).

For each limit function we define its integral to be \( \lim \{ \int \phi_n \} \).

Ian concludes the programme by using the definition of integral in \( L^{\text{inc}} \) to prove the result, which he earlier suggested was intuitively reasonable, that the integral of the characteristic function of any null set is zero. This result follows directly from the fact that such functions are limits almost everywhere of the sequence of zero functions.

Broadcast Self-Assessment Question (BSAQ)

For the function \( f \) in diagram A of the notes above find a simpler sequence \( \{ \phi_n \} \) converging almost everywhere to \( f \) which shows that \( \int f = 0 \).
Solution to BSAQ

The increasing sequence of step functions $\{\phi_n\}$ where

$$\phi_n : x \mapsto 0 \quad (x \in \mathbb{R})$$

converges a.e. to $f$ and so $\int f = \lim \{\int \phi_n\} = 0.$
THE INTEGRAL OF A STEP FUNCTION

A  Introduction

In this programme Fred Holroyd looks at the definition of the Lebesgue integral for step functions and goes through the proof of its consistency in detail.

You will find the programme more helpful if you have read Unit 2.

The notes in Section B are to be followed during the broadcast. In Section C, we give a detailed summary of the programme to be used for revision after the broadcast: it is not intended that you should try to follow this summary during the broadcast.

B  Notes to be followed during the broadcast

1  The characteristic function on a subset $A$ of $\mathbb{R}$

\[ \chi_A(x) = \begin{cases} 
1 & x \in A \\
0 & \text{otherwise.} 
\end{cases} \]

1a  The characteristic function on a bounded interval $A = [a,b]$

\[ \int \chi_A = \text{area of rectangle} = b-a \]
2 A step function is a finite linear combination of characteristic functions of bounded intervals

\[ \phi = a_1 x_{I_1} + a_2 x_{I_2} + \ldots + a_m x_{I_m} \]

3 A specific step function

\[ \phi = 3x_{[1,12]} + 3x_{[5,10]} \]

4 Linearity

\[ f(a_1 x_{I_1} + \ldots + a_m x_{I_m}) = a_1 f x_{I_1} + \ldots + a_m f x_{I_m} \]

5 Another expression for example 3

\[ \phi = 3x_{[1,10]} + 3x_{[5,12]} \]
6 \textit{\text{\textbf{f}} \phi \text{ from expression 3}}

\[ f\phi = 3f_{x[1,12]} + 3f_{x[5,10]} \]
\[ = 3 \times 11 + 3 \times 5 = 48. \]

7 \textit{\textbf{f}} \phi \text{ from expression 5}

\[ f\phi = 3f_{x[1,10]} + 3f_{x[5,12]} \]
\[ = 3 \times 9 + 3 \times 7 = 48. \]

8 \textit{\textbf{A third expression for \phi}}

8a \textit{\textbf{f}} \phi \text{ by expression 8}

\[ f\phi = 3f_{x[1,5]} + 6f_{x[5,10]} + 3f_{x(10,12)} \]
\[ = 3 \times 4 + 6 \times 5 + 3 \times 2 \]
\[ = 12 + 30 + 6 = 48 \]
9  Splitting up the interval \([1, 12]\)

\([1, 12] = [1, 5) \cup [5, 10] \cup (10, 12]\)

10  Deriving expression 8 from expression 3

\[
\phi = 3\chi_{[1, 12]} + 3\chi_{[5, 10]} \quad \text{(expression 3)}
\]

\[
= 3(\chi_{[1, 5)} + \chi_{[5, 10]} + \chi_{[10, 12]}) + 3\chi_{[5, 10]} \quad \text{(using 9)}
\]

\[
= 3\chi_{[1, 5)} + 6\chi_{[5, 10]} + 3\chi_{[10, 12]} \quad \text{(expression 8)}
\]

11  Splitting up the length of \([1, 12]\)

\[\ell([1, 12]) = \ell([1, 5)) + \ell([5, 10]) + \ell([10, 12])\]

12  Proof that expressions 3 and 8 give same \(I\phi\)

\[
I\phi = I(3\chi_{[1, 12]} + 3\chi_{[5, 10]}) \quad \text{(using 3)} \quad \text{_________ (a)}
\]

\[
= 3\ell([1, 12]) + 3\ell([5, 10]) \quad \text{_________ (b)}
\]

\[
= 3(\ell([1, 5)) + \ell([5, 10]) + \ell([10, 12])) + 3\ell([5, 10]) \quad \text{_________ (c)}
\]

\[
= 3\ell([1, 5)) + 6\ell([5, 10]) + 3\ell([10, 12]) \quad \text{_________ (d)}
\]

\[
= I(3\chi_{[1, 5)) + 6\chi_{[5, 12]} + 3\chi_{[10, 12]})) \quad \text{_________ (e)}
\]

(as expression 8)

**Example 13**

\[
\begin{align*}
\phi(x) & \quad 3 \\
2 & \quad \text{ } \\
1 & \quad \text{ } \\
\end{align*}
\]

\[1 \quad 2 \quad 3 \quad 4 \quad -28 -\]

\[x\]
One expression for $\phi$ of 13

\[ \phi = x_{[1,3]} + 2x_{[3,4]} \]

Breaking up 13a into subinterval components

\[ \phi = x_{[1,3]} + 3x_{[3,3]} + 2x_{(3,4)} \]

Consistency for the general case

\[ \phi = c_1x_{I_1} + c_2x_{I_2} + \ldots + c_kx_{I_k} \quad (a) \]

\[ = d_1x_{J_1} + d_2x_{J_2} + \ldots + d_kx_{J_k} \quad (b) \]

Subdivision into all subintervals from both 14(a) and 14(b)

End-points in ascending order

\[ a_1 \quad a_2 \quad \ldots \quad a_{n-1} \quad a_n \]
Set of \(2n-1\) intervals in \(15a\)

\[
\{[a_1, a_1], (a_1, a_2), [a_2, a_2], \ldots, (a_{n-1}, a_n), [a_n, a_n]\}
\]

Let \(m = 2n-1\); call these intervals \(\{K_1, K_2, \ldots, K_m\}\).

Expressing 14(a) and 14(b) in terms of \(\{K_1, K_2, \ldots, K_m\}\)

\[\phi = c_1 x_{I_1} + c_2 x_{I_2} + \ldots + c_k x_{I_k} \quad (14 \text{ (a)})\]

\[= e_1 x_{K_1} + e_2 x_{K_2} + \ldots + e_m x_{K_m} \quad (a)\]

\[\phi = d_1 x_{J_1} + d_2 x_{J_2} + \ldots + d_l x_{J_l} \quad (14 \text{ (b)})\]

\[= f_1 x_{K_1} + f_2 x_{K_2} + \ldots + f_m x_{K_m} \quad (b)\]

C Summary of Programme

Defining the integral of a step function

Fred Holroyd begins the programme by reminding you of the basic ideas of Lebesgue integration. He gives the definition of the integral for characteristic functions of bounded intervals (note 1a) and then goes on to consider the problem of defining the integral of "step functions". A step function is just a finite linear combination of characteristic functions of bounded intervals (notes 2 and 3) and so if we want the integration operator to be linear then the integral of a step function must be given by a corresponding linear combination of the integrals of
its constituent characteristic functions as in note 4. The only problem with using note 4 as a definition is that it may not be consistent; that is, there may be several different ways of expressing a step function in terms of characteristic functions and each may give a different answer for the integral. The purpose of this programme is to show that the definition is in fact consistent.

A Special Case of Consistency

Before looking at the general case Fred Holroyd considers the two different expressions for the step function $\phi$ given in notes 3 and 5 and verifies that they both give the same value for the integral (notes 6 and 7). Geometrically this appears to be obvious for it corresponds to splitting the area under the graph of $\phi$ in different ways and summing the areas. The total area will be the same no matter how the splitting is done.

To see why this happens algebraically it is convenient to introduce a third expression for $\phi$ as in Note 8; this expression is particularly useful because it is defined in terms of disjoint intervals with endpoints that coincide with the end-points of all the intervals in expressions 3 and 5. It is therefore possible to split up the overlapping intervals of expressions 3 and 5 into the disjoint intervals of expression 8 and this result can be used to show that expressions 3 and 8 both give the same step function (see note 10). In Note 12 it is shown that the integral given by expression 3 is the same as that given by expression 8.
In the SAQ at the end of these notes we ask you to deduce analogous results for expressions 5 and 8. It then follows that, since expressions 3 and 5 both give the same value for the integral as expression 8, they must also give the same value as each other, demonstrating the consistency of the definition in this particular case.

The General Case

Returning to the general case, Fred Holroyd considers two general expressions for an arbitrary step function \( \phi \); they are written out in note 14.

Using an argument similar to that given in the special case, we look for a simpler expression for \( \phi \) with intervals that are disjoint subintervals of both the \( I_i \)'s and \( J_i \)'s of expressions 14a and 14b.

The way to do this is illustrated in notes 15 and 16; we take all the end-points of all the intervals involved in expressions 14a and 14b and label them \( a_1, \ldots, a_n \) (Diagram 15a). We then define the intervals \( K_1, \ldots, K_m \) as in Note 16. By construction, these are disjoint intervals and every one of the \( I_i \)'s and \( J_i \)'s is the union of some of these \( K_i \)'s.

It follows that both 14(a) and 14(b) can be expressed in terms of the \( K_i \)'s; the resulting expressions are written out in 17(a) and 17(b) respectively. We are now in a position to prove consistency. We have to prove that expressions 14a and 14b both give the same integral. By an argument similar to that given in note 12, it is easy to show that
both 14a and 17a give the same integral and the same is true for 14b and 17b. Thus, if it can be shown that 17a and 17b have the same integral then the result follows. In fact we can show more than this; we can show that 17a and 17b are one and the same expression. Consider, for example, \( \phi(x) \) where \( x \in K_2 \). Because the \( K_i \)'s are disjoint, \( x \) cannot belong to any other of the \( K_i \)'s and so evaluating \( \phi(x) \) using each of the expressions 17a and 17b in turn gives \( \phi(x) = e_2 = f_2 \). Similarly \( e_i = f_i \), \( i = 1, 2, \ldots, m \). This shows that 17(a) and 17(b) are identical and the consistency of the definition follows.

D Broadcast Self-Assessment Question (BSAQ)

Go through an argument similar to that in notes 9, 10, 11 and 12 of Section B to prove

(a) \( 3x_{[1,10]} + 3x_{[5,12]} = 3x_{[1,5]} + 6x_{[5,10]} + 3x_{(10,12)} \)

(expression 5) \hspace{1cm} (expression 8)

(b) The integral of \( \phi \) (using rule 4: linearity) from expression 5 is the same as from expression 8.
Solution to BSAQ

(a) $3x_{[1,10]} + 3x_{[5,12]} = 3(x_{[1,5]} + x_{[5,10]}) + 3(x_{[5,10]} + x_{[10,12]})$

$= 3x_{[1,5]} + 6x_{[5,10]} + 3x_{[10,12]}$

(b) $\int \phi = \int (3x_{[1,10]} + 3x_{[5,12]})$ (using expression 5)

$= 3x_{[1,10]} + 3x_{[5,12]}$

$= 3(x_{[1,5]} + x_{[5,10]}) + 3(x_{[5,10]} + x_{[10,12]})$

$= 3x_{[1,5]} + 6x_{[5,10]} + 3x_{[10,12]}$

This last line is the integral of $\phi$ using expression 8.
TELEVISION PROGRAMME 3

WHEN IS A FUNCTION INTEGRABLE?

Introduction to the Programme

In this programme Allan Solomon and Ian Dey give some simple, sufficient criteria for deciding whether a function is Lebesgue integrable. They examine why these criteria work, and discuss how functions satisfying them can be integrated. You may find it helpful to have read the introduction to Unit 4, *Definite and Indefinite Integrals* before viewing the programme.

Summary of the Programme

*Three Conditions*

The four stage process used to construct the Lebesgue integral is often too cumbersome to help us decide whether or not a function is integrable. We would like to find a straightforward way of deciding whether a function belongs to $L^1$.

Fortunately, many functions which interest us satisfy the following conditions which are sufficient (but not necessary) for a function to be Lebesgue integrable.

(i) $f$ vanishes outside a closed bounded interval $[a,b]$;

(ii) $f$ is bounded (i.e. there is a real number $K$ such that $-K \leq f(x) \leq K$ for all $x \in \mathbb{R}$);
(iii) \( f \) is continuous almost everywhere.

In fact functions satisfying these conditions are not only Lebesgue integrable but also elements of \( \text{L}^{\text{inc}} \).

\textit{An Example}

We illustrate the sufficiency of conditions (i) - (iii) using the function \( f \) whose graph is

\[
\begin{array}{c}
\text{It is easy to see that } f \text{ satisfies all the conditions (i) - (iii). We show that } f \text{ belongs to } \text{L}^{\text{inc}} \text{ by constructing an increasing sequence } \{ \phi_n \} \text{ of step functions with bounded integrals, converging almost everywhere to } f. \text{ The construction we use resembles the construction of lower sums in Riemann integration. } \phi_1 \text{ is defined on the partition of } [a,b] \text{ obtained}
\end{array}
\]
by bisecting \([a, b]\); we construct the largest step function on this partition that fits under the curve:

\[ \phi_1 \quad \phi_2 \]

\[ a \quad b \quad a \quad b \]

\( \phi_2 \) is defined in a similar way on the partition of \([a, b]\) obtained by bisecting each interval in the partition used for \( \phi_1 \).

Continuing this process of bisection we obtain a sequence of step functions \( \{ \phi_n \} \). At each stage the functions \( \phi_n \) cannot decrease and, furthermore, the sequence of integrals \( \{ \int \phi_n \} \) must be bounded by the area of the large rectangle. We show later that \( \{ \phi_n \} \) converges to \( f \) almost everywhere.

**The General Case**

We now show that the same construction works for any function satisfying conditions (i) - (iii). At each step of the bisection procedure we can always find a largest step function lying under the graph of \( f \) because \( f \) is bounded below, and so on each interval of a partition we can define the value of the step function to be the infimum of \( f \) over that interval. The resulting sequence of step functions \( \{ \phi_n \} \) must be bounded by \( K \times (b-a) \). It remains to show that \( \{ \phi_n \} \)
converges a.e. to \( f \); in fact we show that \( \{\phi_n(p)\} \) converges to \( f(p) \) at all points \( p \) where \( f \) is continuous. Consider a point \( p \) at which \( f \) is continuous. For any \( \varepsilon > 0 \) we want to find an \( N \) such that for all \( n \geq N \), \( \phi_n(p) \) lies within \( \varepsilon \) of \( f(p) \). Now for each \( n \), let \( A_n \) be that interval of the \( n \)th partition which contains \( p \). Then \( \phi_n(p) \) is by definition equal to the infimum of \( f(A_n) \). But since \( f \) is continuous we can choose a \( \delta \) such that the image of all numbers within \( \delta \) of \( p \) lies within \( \varepsilon \) of \( f(p) \). Hence by choosing \( N \) large enough to ensure that the length of \( A_N \) is less than \( \delta \), the set \( f(A_n) \) along with its infimum, \( \phi_N(p) \), will lie within \( \varepsilon \) of \( f(p) \). Furthermore, for all \( n \geq N \), \( \phi_N(p) \leq \phi_n(p) \leq f(p) \) and so \( \phi_n(p) \) will also lie within \( \varepsilon \) of \( f(p) \), as required.

Having shown that functions satisfying conditions (i) – (iii) are Lebesgue integrable, we produce counter-examples to show that the converse statement is not true. Consider the function \( \chi_Q \) which is Lebesgue integrable. \( \chi_Q \) is bounded but fails to satisfy conditions (i) and (iii). There again, consider the function

\[
\begin{align*}
f : x \mapsto \begin{cases} 
\frac{1}{\sqrt{x}} & \text{if } x \in (0,1) \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

\( f \) is Lebesgue integrable but does not satisfy condition (ii). The sum \( \chi_Q + f \) is a Lebesgue integrable function which satisfies none of the conditions.
Evaluating the Integral

Although conditions (i) - (iii) are not necessary for Lebesgue integration, they are both necessary and sufficient for Riemann integration. It follows that the space of Riemann integrable functions is just a subset of $L^1$; furthermore, it can be shown that the Riemann integral and the Lebesgue integral coincide on this set. Thus having used the conditions (i) - (iii) to decide that a function is Lebesgue integrable we can then use all the old techniques of Riemann integration to evaluate its integral.

Broadcast Self-Assessment Question (BSAQ)

(i) Which of the functions below satisfy the three conditions mentioned in the programme?

A: $x \mapsto \cos x$ (xeR)

B: $x \mapsto x_1(x)$ (xeR) where $I$ is the set, $[0,1] \setminus Q$, of irrational numbers in $[0,1]$

\[
\begin{cases}
\frac{1}{x} & \text{if } x \in [-2,-1] \\
x & \text{if } x \in [-1,1] \\
0 & \text{otherwise}
\end{cases}
\]

C: $x \mapsto \begin{cases}
\frac{1}{x} & \text{if } x \in (0,10] \\
0 & \text{otherwise}
\end{cases}$

D: $x \mapsto \begin{cases}
\frac{1}{x} & \text{if } x \in [-10,10] \\
0 & \text{otherwise}
\end{cases}$

E: $x \mapsto \begin{cases}
e^{-x}\sin 3x & \text{if } x \in [-10,10] \\
0 & \text{otherwise}
\end{cases}$

(ii) Which of the functions in (i) are Lebesgue integrable?
Solutions to BSAQ

(i) A does not satisfy the conditions; it does not vanish outside a bounded interval.

B does not satisfy the conditions; it is not continuous almost everywhere.

C satisfies all three conditions.

D does not satisfy the conditions; it is not bounded.

E satisfies all the conditions.

(ii) Functions B, C and E are Lebesgue integrable.
RADIO PROGRAMME 3

NULL SETS

A Introduction to the Programme
Allan Solomon talks about the role of null sets in the theory of Lebesgue integration. He first considers the case of functions defined on \( \mathbb{R} \) and then indicates how the same ideas can be generalized to functions defined on \( \mathbb{R}^k \).

You will find the programme easier to follow if you have read the introduction to Unit 6, *The Lebesgue Integral on \( \mathbb{R}^k \).*

The notes in Section B are to be following during the broadcast. In section C, we give a detailed summary of the programme to be used for revision after the broadcast; it is not intended that you should try to follow this summary during the broadcast.

B Notes to be followed during the broadcast

1 Null subset \( N \) of \( \mathbb{R} \)
   A subset \( N \) of \( \mathbb{R} \) is null if it can be covered by a sequence of (bounded) open intervals whose total length is arbitrarily small.

2 Any countable subset of \( \mathbb{R} \) is null.
Cantor's Ternary Set $S = \lim_{n \to \infty} \{ S_n \}$ (i.e. $\bigcup_{n=1}^{\infty} \{ S_n \}$)

\[ \begin{array}{cccc}
0 & \frac{1}{3} & \frac{2}{3} & 1 \\
0 & \frac{1}{9} & \frac{2}{9} & \frac{1}{3} & \frac{2}{3} & \frac{7}{9} & \frac{8}{9} & 1
\end{array} \]

Union of $2^n$ closed intervals, each of length $\left(\frac{1}{3}\right)^n$.

TOTAL LENGTH = $\left(\frac{2}{3}\right)^n$

4 \textit{Linc} \textit{(Stage 3 of the 4-stage construction of Lebesgue integral)}

\[ f(x) = \lim_{n \to \infty} \{ \phi_n(x) \} \]

\[ [\{ \phi_n \} \downarrow \text{ and } \{ f \phi_n \} \text{ bd.}] \]

define $f = \lim \{ f \phi_n \}$

5 \textit{Theorem:}

\[ [\{ \phi_n \} \downarrow \text{ and } \{ f \phi_n \} \text{ bd.}] \Rightarrow [\phi_n \downarrow f \text{ a.e.}] \]
6 Sufficient condition for integrability

\[ f = 0 \text{ outside a bounded interval} \]

\[ f \text{ bounded} \quad \implies \quad f \text{ integrable} \]

\[ f \text{ continuous a.e.} \]

\[ (\text{its points of discontinuity form a null set}) \]

7 The Lebesgue Integral on \( \mathbb{R}^k \)

\[ \int : (f: \mathbb{R}^k \to \mathbb{R}) \to \mathbb{R} \]
8 Bounded interval $K$ of $\mathbb{R}^2$

9 Stage 1 - The integral of a characteristic function of a bounded interval
Null subset \( N \) of \( \mathbb{R}^2 \)

A subset \( N \) of \( \mathbb{R}^2 \) is null if it can be covered by a sequence of (bounded) open intervals whose total area is arbitrarily small.

Null subset \( N \) of \( \mathbb{R}^k \)

A subset \( N \) of \( \mathbb{R}^k \) is null if it can be covered by a sequence of (bounded) open intervals whose total measure is arbitrarily small.

Area of a set \( S \subset \mathbb{R}^2 \) is \( \mathcal{A}_S \) (if \( \mathcal{A}_S \) exists)

Finite Polygon \( S \subset \mathbb{R}^2 \)

C. Summary of the Programme

Null sets in \( \mathbb{R} \)

The definition of a null set is given in Section B, Note 1. A more technical definition is given in Unit 1, The Real Numbers.

Examples of null sets are easy to find; for example, it is shown in Weir: page 18 that all countable sets are null (Note 2). An example of a null set which is not countable is Cantor's Ternary Set (Note 3). For a discussion of Cantor's set see Weir: page 20.
Null Sets and Integration Theory

The importance of null sets stems primarily from the fact that the Lebesgue integral of a function \( f \) is independent of the behaviour of \( f \) on a null set.

Null sets first come into Lebesgue integration theory in the third stage of the four stage construction of the integral where we construct \( L^1 \). There we define the integral of functions which are limits almost everywhere (i.e. limits everywhere except possibly on a null set) of sequences of step functions with bounded integrals (note 4). The reason we use convergence almost everywhere in the definition is explained by the theorem written out in Note 5; a sequence of step functions with bounded integrals may not converge everywhere.

Null sets, having been introduced, continue to occur throughout the development of the theory. For example, one criterion for a function \( f \) to be integrable is that (1) \( f \) is bounded, (2) \( f \) vanishes outside a bounded interval, and (3) \( f \) is continuous almost everywhere (note 6). It does not matter if \( f \) is discontinuous on some null set. This illustrates an important point concerning integration theory that the behaviour of a function on a null set can often be ignored.

Null sets in \( \mathbb{R}^k \)

Null sets can be defined on sets other than \( \mathbb{R} \). They can, for example, be defined on the product sets \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) or, more
generally, on \( \mathbb{R}^k \). Indeed, we will meet the more general concept of null set when in Unit 6, *The Lebesgue Integral on \( \mathbb{R}^k \)*, the Lebesgue integral is extended to functions \( f: \mathbb{R}^k \rightarrow \mathbb{R} \) with domain \( \mathbb{R}^k \) (Note 7). The construction of the integral on \( \mathbb{R}^k \) follows the familiar 4 stage process used for \( \mathbb{R} \) with one or two changes. The changes that have to be made can be understood by considering the case \( \mathbb{R}^2 \) (\( k = 2 \)). In the first stage of the construction we still consider characteristic functions of "bounded intervals" but, whereas in \( \mathbb{R} \) bounded intervals are sets of the form \([a, b]\), in \( \mathbb{R}^2 \) bounded intervals are sets of the form \([a, b] \times [c, d]\) (Notes 8 and 9). Another change is in stage three of the construction. We again require the concept of convergence almost everywhere, but for this we require a definition of null set in \( \mathbb{R}^2 \). This definition of a null set is given in note 10. It differs from the definition given for \( \mathbb{R} \) only in that it talks of the total area rather than the total length of the covering intervals.

In the general case of \( \mathbb{R}^k \) we measure the size of an interval by defining the "measure" of an interval (*Weir: page 71*). The term "measure" is just a generalization of length, area, etc., and enables us to make the general definition of null set in \( \mathbb{R}^k \) given in Note 11.

Apart from these two changes, the construction proceeds in exactly the same way as for \( \mathbb{R} \) and it is not surprising that theorems concerning the Lebesgue integral on \( \mathbb{R} \) carry over virtually unchanged to \( \mathbb{R}^k \). In particular, a function which is bounded, zero
outside a bounded interval of $\mathbb{R}^k$, and continuous on all but a null set of $\mathbb{R}^k$, is necessarily integrable. This theorem again illustrates the point that the behaviour of functions on null sets can often be ignored.

A Finite Polygon has an area

A finite polygon is a subset of $\mathbb{R}^2$, bounded by a finite number of straight lines, which can be contained in a bounded interval (Note 13). Suppose $S$ is a finite polygon. $S$ is said to have an area if the characteristic function $\chi_S$ is integrable (Note 12). By the theorem mentioned above, $\chi_S$ is integrable since it vanishes outside a bounded interval, it is bounded by 1, and the points of discontinuity are on the boundary line which forms a null set. The result follows.
Broadcast Self-Assessment Questions (BSAQ's)

1. Let $S$ be the set of real numbers in the closed interval $[0, 1]$ which have a binary expansion with zero's in all the even places (e.g. $0.1000100010 \ldots$). Show that $S$ is a null set.

2. Show that $Q \times R$ is a null set of $R^2$.
   (Hint: Any straight line in $R^2$ is a null set)

3. Show that $N \times R$ is a null set of $R^2$, where $N$ is a null set of $R$.
   (Hint: $N \times R$ is a union of sets of the form $N \times (m, m+1]$ where $m \in \mathbb{Z}$.)
Solutions to BSAQ's

1. Numbers in $S$ have a zero in the 2nd place of their binary expansions and are therefore contained in the unshaded intervals

$$
\begin{array}{cccccc}
0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\
\end{array}
$$

of total length $\frac{1}{2}$. The numbers in $S$ also have a zero in the 4th place of the binary expansion and are contained in the unshaded intervals

$$
\begin{array}{cccccccc}
0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & \frac{1}{8} & \frac{3}{8} & \frac{5}{8} & \frac{7}{8} & 1 \\
\end{array}
$$

of total length $\frac{1}{4}$. Similarly by considering $2^n$ th place of the binary expansions we see that $S$ can be covered by intervals of total length $\left(\frac{1}{2}\right)^n$. Since $\lim \left\{\left(\frac{1}{2}\right)^n\right\} = 0$, $S$ is a null set.

2. $Q \times R$ can be written as $\bigcup_{q \in Q} \{q\} \times R$. Now for each $q$, $\{q\} \times R$ is a straight line in $R^2$ and hence a null set. $Q \times R$ is therefore null since it is a countable union of null sets.

3. $N \times R$ can be written as $\bigcup_{m \in \mathbb{Z}} N \times (m,m+1]$. Choose any $\varepsilon > 0$ and let $N$ be covered by a sequence of intervals $\{I_n\}$ such that $\sum_{n=1}^{\infty} \ell(I_n) \leq \varepsilon$. Then for any $m \in \mathbb{Z}$, $N \times (m,m+1]$ is covered by the sequence of intervals $\{I_n \times (m,m+1]\}$ whose total area is $\sum_{n=1}^{\infty} \ell(I_n) \times \ell((m,m+1]) = \sum_{n=1}^{\infty} \ell(I_n) \leq \varepsilon$, $N \times R$ is therefore a countable union of null sets and hence null.
THE LEBESGUE INTEGRAL ON $\mathbb{R}^k$

Introduction to the programme
Allan Solomon and Ian Dey discuss the extension of Lebesgue integration to functions of several variables. In the first half of the programme they describe the construction of the integral. They then introduce and look at the proof of an important theorem called Fubini's theorem, an extremely useful tool for evaluating integrals.

Summary of the Programme

Construction of the integral
The construction of the Lebesgue integral on $\mathbb{R}^k$ is a straightforward generalization of the four stages used for the integral on $\mathbb{R}$. In the programme we consider the case $\mathbb{R}^2$. In the first stage of the construction we define the integral of characteristic functions of "bounded intervals". Bounded intervals of $\mathbb{R}^2$ are "rectangles", that is, sets of the form $I \times J$, where $I$ and $J$ are bounded intervals of $\mathbb{R}$. The integral of $\chi_{I \times J}$ is defined to be $\ell(I) \times \ell(J)$ and corresponds to the volume under the surface representing $\chi_{I \times J}$. Stage two, the extension to step functions, proceeds from stage one in exactly the same way as for $\mathbb{R}$. 

- 51 -
For the third stage we define elements of $L^\text{inc}$ to be functions which are limits "almost everywhere" of increasing sequences of step functions with bounded integrals. Before doing this, however, we must give meaning to the expression _almost everywhere_ by defining null sets on $\mathbb{R}^2$. By analogy with $\mathbb{R}$ null sets are defined to be sets that can be covered by a sequence of open intervals whose total area is arbitrarily small. Examples of null sets include straight lines, straight line segments and points. The integral of a function in $L^\text{inc}$ is defined to be the limit of the integrals of an increasing sequence of step functions converging almost everywhere to the function concerned.

The fourth stage converts $L^\text{inc}$ into a vector space by taking differences of functions in $L^\text{inc}$.

_Evaluating the Integral_

The evaluation of integrals on $\mathbb{R}^k$ is greatly simplified by a useful theorem called Fubini's theorem which relates integrals on $\mathbb{R}^k$ to integrals on lower dimensions. In the case of $\mathbb{R}^2$ the statement of Fubini's theorem can be summarized in classical notation by

$$\int f(x,y) \, d(x,y) = \int \left( \int f(x,y) \, dy \right) \, dx.$$ 

Thus to integrate $f$ we keep $x$ fixed and integrate with respect to $y$; this gives a function of $x$ whose integral is equal to $\int f$. 

- 52 -
We illustrate this process with the aid of a model. We first integrate each cross-section \((x = \text{constant})\) of \(f\). In this way we assign a real number to each \(x\) thereby defining a function \(F\) whose integral is equal to \(\int f\). Clearly this process depends on our being able to integrate each cross-section of \(f\); in general, however, this is not always possible. Consider any function \(g\) which coincides with an integrable function \(f\) everywhere except on the line \(x = 5\) where it takes the value 1. Now \(g\) must be integrable but the cross-section of \(g\) at \(x = 5\) is \(\chi_{\mathbb{R}}\) which is not integrable. In fact this is not a serious problem because the precise statement of Fubini's theorem only requires the cross-section \(f_x\) to be integrable for *almost all* \(x\). Fubini's theorem states if \(f: \mathbb{R}^2 \to \mathbb{R}\) is integrable, the cross-section

\[
f_x : y \mapsto f(x,y)
\]

is integrable for almost all \(x\), and

\[
\int f(x,y) dy \overset{a.e.}{=} F(x) \text{ for some } F \in L^1;
\]

furthermore,

\[
\int f(x,y) d(x,y) = \int F(x) dx.
\]

Notice that the final equation is independent of the "bad" values of \(x\) for which \(f_x\) is not integrable.
Proof of Fubini's Theorem

The proof of Fubini's theorem follows the four stages used to construct the integral. It is easy to prove the theorem for characteristic functions of bounded intervals for

\[ \int \chi_{I \times J}(x,y) \, d(x,y) = \ell(I) \times \ell(J) = \int \left( \int \chi_{I \times J} \, dy \right) \, dx, \]

and we show that the theorem holds for step functions by linearity. The most difficult part of the proof is the extension to \( L^{\text{inc}} \). We do not attempt to work through this part of the proof in the programme but indicate the steps involved with the following commutative diagram.

To complete the proof we have to show that the theorem works for all functions in \( L^1(\mathbb{R}^2) \). This result follows from \( L^{\text{inc}} \) by linearity.
Broadcast Self-Assessment Question (BSAQ)

In the BSAQ we consider the commutative diagram given at the end of the broadcast notes. We use it to work through the proof of the third stage of Fubini's theorem as given in Weir. We suggest you leave this question until you reach the relevant section of Unit 7.

We are given that \( f \) belongs to \( L^\text{inc} \). Hence there is an increasing sequence of step functions \( \{ \phi_n \} \), whose integrals are bounded, converging almost everywhere to \( f \). Furthermore, \( \int f = \lim \{ \int \phi_n \} \). (This corresponds to the grey rectangle of the commutative diagram.)

So far we have proved Fubini's theorem for step functions and so for each \( \phi_n \) we have:

The cross-section \( \phi_n : y \mapsto \phi_n(x,y) \) (we shall sometimes write \( \{ \phi_n \}_x \) for the sequence \( \{ \phi_n \}_x \)) is a step function, and there is a step function \( \phi_n \) such that

\[
\int \phi_n(x,y)dy = \phi_n(x) \quad \text{for all } x
\]

and

\[
\int \phi_n(x,y)d(x,y) = \int \phi_n(x)dx.
\]

(This corresponds to the bottom triangle of the commutative diagram.)
The proof of the third stage of Fubini's theorem involves proving three things:

(i) The cross-section $f_x^y : y \mapsto f(x, y)$ is integrable for almost all $x$.

(ii) There is an integrable function $F : \mathbb{R} \to \mathbb{R}$ such that

$F(x) = \int_{f_x^y} f(y) dy$ for almost all $x$.

(iii) $\int f(x, y) d(x, y) = \int F(x) dx$.

(This corresponds to the top triangle of the commutative diagram.)

We proceed as follows:

(a) Using Lemma 4.3.1 (Weir: page 88) show that, for almost all $x$,

the cross-section $f_x^y$ is the limit almost everywhere of the sequence of step functions $\{ \phi_{n, x} \}$. Show that $\{ \phi_{n, x} \}$ is increasing.

(Let $S_x^y$ denote the set of $y$'s in $\mathbb{R}$ for which $\{ \phi_{n, x} \}$ does not converge to $f_x^y$.)

(b) We know, using the result for step functions, that $\int f = \lim \{ \int \phi_n \} = \lim \{ \int \phi_n \}$. Why does this imply that $\{ \phi_n \}$ is convergent almost everywhere? (Let $B$ denote the set of numbers $x$ for which $\{ \phi_n \}$ diverges.)

(c) Define

$$F(x) = \begin{cases} 
\lim \{ \phi_n (x) \} & \text{for } x \in A \cup B \\
0 & \text{otherwise.} 
\end{cases}$$

Show that $f_x^y$ is integrable and that $\int f_x^y = F(x)$ for all $x \in A \cup B$ (i.e. for almost all $x$). (This shows that the black rectangle of the commutative diagram is commutative.)
(d) Use $\int f = \lim \{ \int \phi_n \}$ to prove that $\int f = \int F$. (This shows that the white rectangle of the commutative diagram is commutative.)

This completes the third stage of the proof.

**Solution to BSAQ**

(a) $\{ \phi_n \}$ converges to $f$ everywhere except on a null set $S \subset \mathbb{R}^2$. Thus, $\{ \phi_n(y) \}_x$ converges to $f_x(y)$ for all $y$ not in the set $S_x = \{ y : (x, y) \in S \}$. But, by Lemma 4.3.1, $S_x$ is a null set of $\mathbb{R}$ and so $\{ \phi_n \}_x$ converges to $f_x$ almost everywhere.

$\{ \phi_n \}_x$ is increasing because $\{ \phi_n \}$ is increasing.

(b) This is a direct consequence of Theorem 4 (Weir: page 77). $\{ \phi_n \}$ is an increasing sequence of step functions on $\mathbb{R}$, whose integrals converge to $\int f$. Therefore $\{ \phi_n \}_x$ converges a.e. on $\mathbb{R}$.

(c) Consider any $x \notin A \cup B$. Then $f_x$ is the limit almost everywhere of the increasing sequence of step functions $\{ \phi_n \}_x$. Furthermore, the sequence of integrals $\{ \int \phi_n \}_x = \{ \phi_n(x) \}$ converges (and is therefore bounded). It follows that $f_x$ is integrable and that

$$\int f_x = \lim \{ \int \phi_n \}_x = \lim \{ \phi_n(x) \} = F(x) \text{ for all } x \notin A \cup B.$$  

(d) $\phi_n$ is an increasing sequence (since $\{ \phi_n \}$ is increasing) of step functions converging almost everywhere to $F$. The sequence of integrals $\{ \int \phi_n \}$ is bounded (by $\int f$). It follows that

$$\int F = \lim \{ \int \phi_n \} = \int f.$$
FUBINI'S THEOREM

A Introduction

In this programme Ian Dey describes the way in which Fubini's theorem is used to evaluate the integral of a function of several variables. He illustrates the process by integrating two functions in $L^1(R^2)$.

You will find the programme more helpful if you have read the introduction to Unit 7.

The notes in Section B are to be following during the broadcast. In section C, we give a detailed summary of the programme to be used for revision after the broadcast: it is not intended that you should try to follow the summary during the broadcast.
B Notes to be followed during the broadcast

1a Integrable \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \)
\[
(x,y) \mapsto f(x,y)
\]

1b \( f_x : \mathbb{R} \rightarrow \mathbb{R} \)
\[
y \mapsto f(x,y)
\]

1c \( F : \mathbb{R} \rightarrow \mathbb{R} \)
\[
F(x) = \int f_x \quad \text{(i.e. } \int f_x(y)dy)\]

1d \( \int F \quad \text{(i.e. } \int F(x)dx)\)

2 Fubini's Theorem

If \( f \) is integrable, then \( \iint f = \int F \) (i.e. \( \int F(x)dx \)).

3 Fubini's Theorem in classical notation

as in 1c
\[
\int f(x,y)d(x,y) = \int \left( \int f(x,y)dy \right)dx
\]

as in 1d
\[
= \int \left( \int f(x,y)dx \right)dy
\]
4a Example 1

\[ f(x, y) = \begin{cases} 
    x^2 + y^2 & \text{if } (x, y) \in A \\
    0 & \text{otherwise} 
\end{cases} \]

4b Graph of \( f \)

4c Cross-section at \( x \) parallel to \( y \)-axis
4d Computing $f_x : R \rightarrow R$

\begin{enumerate}
\item \textbf{Case 1} \hspace{1cm} x<1 \text{ or } x>1
\begin{align*}
f_x(y) &= 0 \text{ for all } y \in R
\end{align*}
\item \textbf{Case 2} \hspace{1cm} -1 \leq x \leq 1
\begin{align*}
f_x(y) &= \begin{cases} 
x^2 + y^2 & \text{if } -1 \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\end{enumerate}

4e Computing $F(x) = \int f_x(y) dy$

\begin{enumerate}
\item \textbf{Case 1} \hspace{1cm} x<1 \text{ or } x>1
\begin{align*}
F(x) &= \int f_x(y) dy = \int 0 dy = 0
\end{align*}
\item \textbf{Case 2} \hspace{1cm} -1 \leq x \leq 1
\begin{align*}
F(x) &= \int_{-1}^{x} f_x(y) dy = \int_{-1}^{1} x^2 + y^2 dy \\
&= [x^2 y + \frac{1}{3} y^3]_{-1}^{1} = 2x^2 + \frac{2}{3}
\end{align*}
\end{enumerate}

\begin{align*}
\therefore \hspace{1cm} F(x) &= \begin{cases} 
2x^2 + \frac{2}{3} \text{ if } x \in [-1,1] & \text{(case 2)} \\
0 & \text{otherwise} \hspace{1cm} \text{(case 1)}
\end{cases}
\end{align*}

4f \begin{align*}
\int_{-1}^{1} F(x) dx &= \int_{-1}^{1} (2x^2 + \frac{2}{3}) dx = [\frac{2}{3} x^3 + \frac{2}{3} x]_{-1}^{1} = \frac{8}{3}
\end{align*}

\begin{align*}
= \int f \text{ by Fubini's Theorem}.
\end{align*}
5a Example 2

\[ f(x,y) = \begin{cases} 
  x^2 + y^2 & \text{if } (x,y) \in A \\
  0 & \text{otherwise}
\end{cases} \]

5b Computing \( f_x : \mathbb{R} \to \mathbb{R} \)

<table>
<thead>
<tr>
<th>Case 1</th>
<th>( x &lt; 0 \text{ or } x &gt; 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_x(y) = 0 ) for all ( y \in \mathbb{R} )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case 2</th>
<th>( 0 \leq x \leq 2 )</th>
</tr>
</thead>
</table>
| \( f_x(y) = \begin{cases} 
  x^2 + y^2 & \text{if } -\frac{x}{2} \leq y \leq \frac{x}{2} \\
  0 & \text{otherwise}
\end{cases} \) |
5c Computing \( F(x) = \int f_x(y) \, dy \)

\[
\text{Case 1} \quad x<0 \text{ or } x>2
\]

\[
F(x) = \int f_x(y) \, dy = \int_0^x dy = 0
\]

\[
\text{Case 2} \quad 0 \leq x \leq 2
\]

\[
F(x) = \int f_x(y) \, dy = \int_0^x \frac{y^2}{x^2} \, dy
\]

\[
= \left[ \frac{x^2}{2} y + \frac{1}{3} y^3 \right]_{y=0}^{y=x} = \frac{13}{12} x^3
\]

\[
\therefore \quad F(x) = \begin{cases} 
\frac{13}{12} x^3 & \text{if } x \in [0, 2] \text{ (Case 2)} \\
0 & \text{otherwise} \text{ (Case 1)}
\end{cases}
\]

5d \[
\int F(x) \, dx = \int_0^2 \frac{13}{12} x^3 \, dx = \left[ \frac{13}{48} x^4 \right]_0^2 = \frac{13}{3}
\]

\[
= \int f \text{ by Fubini's Theorem}
\]
6a \textbf{For function } f \textbf{ of } 4a : \int f = \int_{-1}^{+1} \left( \int_{-1}^{+1} x^2 + y^2 \, dy \right) dx

= \int_{-1}^{+1} \left( 2x^2 + \frac{2}{3} x \right) dx

= \left[ \frac{2}{3} x^3 + \frac{2}{3} x^2 \right]_{-1}^{+1} = \frac{8}{3}

6b \textbf{For function } f \textbf{ of } 5a : \int f = \int_{0}^{2} \left( \int_{\frac{x}{2}}^{x} x^2 + y^2 \, dy \right) dx

= \int_{0}^{2} \left( \frac{12}{12} x^3 \right) dx

= \left[ \frac{13}{48} x^4 \right]_{0}^{+1} = \frac{13}{3}

6c

\int f(x,y) d(x,y) = \int_{a}^{b} \left( \int_{y_1(x)}^{y_2(x)} f(x,y) \, dy \right) dx

\text{Region A (outside which } f \text{ is zero)}
\[ \int f(x, y) d(x, y) = \int_{c}^{d} \left( \int_{x_1(y)}^{x_2(y)} f(x, y) dy \right) dy \]

C Summary of Programme

Fubini's Theorem

In Unit 6, The Lebesgue Integral on \( R^k \), the Lebesgue integral is generalized to cope with functions of several variables (i.e. functions from \( R^k \) to \( R \)). Fubini's theorem enables us to evaluate the integral of a function of several variables using techniques developed for functions of one variable.

We illustrate the use of Fubini's theorem by considering functions on \( R^2 \). The steps involved are summarized in Note 1; given an integrable function \( f: R^2 \rightarrow R \) (note 1a), we carry out the following procedure.
1. For each real number \( x \), construct a function \( f_x : \mathbb{R} \to \mathbb{R} \) whose value, \( f_x(y) \), at the point \( y \) in \( \mathbb{R} \) is defined to be \( f(x,y) \) (Note 1b).

2. Construct the function \( F : \mathbb{R} \to \mathbb{R} \) whose value \( F(x) \) at the real number \( x \) is defined to be \( \int f_x \) (Note 1c).

3. Integrate \( F \).

Fubini's theorem states that \( \int f = \int F \). The process in 2 and 3 involves integrating functions of one variable; it is often described by saying "integrate \( f \) with respect to \( y \) keeping \( x \) fixed, and then integrate with respect to \( x \)." This shorthand description is best expressed in terms of Leibniz classical notation (Note 3). For the purpose of evaluating integrals it is often helpful to think of the process in terms of Leibniz notation and to use the notation to carry out the calculation.

**First Example**

Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) defined in Note 4a. A glance at the graph of \( f \) (Note 4b) reveals that \( f \) satisfies the conditions of Theorem 4 (Weir: page 81) and that \( f \) is therefore integrable. It follows that \( \int f \) can be evaluated using Fubini's theorem as indicated above.

1. The functions \( f_x \) are specified in Note 4d (their graphs correspond to cross-sections through the graph of \( f \) parallel to the \( y \)-axis at \( x \) (Note 4c)).
2. The function $F$ is obtained by integrating $f_x$ for each $x$ (Note 4e).

3. By Fubini's theorem, $\int f = \int F = \frac{8}{3}$ (Note 4f).

**Second Example**

As a second example we consider the function $f$ defined in Note 5a. This is similar to the function used in Example 1 but the region $A$ on which $f$ takes non-zero values is triangular instead of square. The evaluation of $\int f$ using Fubini's theorem not surprisingly resembles the calculation in Example 1; it is written out in Notes 5b, 5c and 5d. One difference is that finding $F(x)$ involves evaluating the integral $\int_{\frac{x}{2}}^{\frac{x}{2}} f_x$ (Note 5c). Here the end-points of integration depend on $x$ whereas in Example 1 the corresponding end-points are constant. Geometrically this is because the region $A$ no longer has edges parallel to the $x$-axis.

Examples 1 and 2 are written out using Leibniz notation in Notes 6a and 6b respectively.

**A General Case**

The general case of a function $f$ which vanishes outside a bounded region $A$ is illustrated in Note 6c. The Leibniz formulation of
Fubini's theorem given in Note 6 is easily remembered because the end-points of integration are readily determined by examining the boundary of the region $A$. It is worth noticing that Fubini's theorem is symmetrical in the sense that we can equally well integrate $f$ with respect to $x$ first, keeping $y$ fixed, and then with respect to $y$ (Note 7).

D. **Broadcast Self-Assessment Questions (BSAQ's)**

1. Compute the integral of the function $f: \mathbb{R}^2 \to \mathbb{R}$ where

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } (x,y) \in A \\ 0 & \text{otherwise} \end{cases}$$

2. Compute the integral of the function used in Example 2 (Note 5a) in the reversed order (Note 7).
Solutions to BSAQ's

1. As in note 6c,

\[
\int_a^b \left( \int_{y_1(x)}^{y_2(x)} (x^2 + y^2) \, dy \right) \, dx
\]

\[a = -1, \quad y_1(x) = -\sqrt{1-x^2},\]
\[b = +1, \quad y_2(x) = +\sqrt{1-x^2},\]

\[
= \int_{-1}^1 \left( \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2) \, dy \right) \, dx
\]

\[
= \int_{-1}^1 \frac{2}{3} (1+2x^2) \sqrt{1-x^2} \, dx
\]

Substituting \( x = \sin \theta \), then \( dx = \cos \theta d\theta \) and \( \sin \frac{\pi}{2} = 1 \), \( \sin -\frac{\pi}{2} = -1 \); thus

\[
\int f = \frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+2 \sin^2 \theta) \cos^2 \theta d\theta
\]

\[
= \frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\cos 2\theta + 1}{2} \right) + \frac{1}{2} \sin^2 2\theta \, d\theta
\]

\[
= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\theta + \frac{1+\cos 4\theta}{2}) \, d\theta
\]

\[
= \frac{1}{3} \left[ \frac{3}{2} \theta - \frac{\sin 2\theta}{2} - \frac{1}{2} \sin 4\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
\]

\[
= \frac{\pi}{2}
\]
From the diagram we see that

\[
\int f = \int_{-1}^{+1} \left( \int_{2|y|}^{2} x^2 + y^2 \, dx \right) \, dy
\]

\[
= \int_{-1}^{+1} \left[ \frac{1}{3} x^3 + 2xy^2 \right]_{2|y|}^{2} \, dy
\]

\[
= \int_{-1}^{+1} \frac{8}{3} + 2y^2 - \frac{14}{3} |y|^3 \, dy
\]

\[
= \int_{-1}^{0} \frac{8}{3} + 2y^2 + \frac{14}{3} y^3 \, dy + \int_{0}^{1} \frac{8}{3} + 2y^2 - \frac{14}{3} y^3 \, dy
\]

\[
= \left[ \frac{8}{3} y + \frac{2}{3} y^3 + \frac{7}{6} y^4 \right]_{0}^{1} + \left[ \frac{8}{3} y + \frac{2}{3} y^3 - \frac{7}{6} y^4 \right]_{0}^{1}
\]

\[
= \left( \frac{8}{3} + \frac{2}{3} - \frac{7}{6} \right) + \left( \frac{8}{3} + \frac{2}{3} - \frac{7}{6} \right)
\]

\[
= \frac{13}{3}
\]
THE LEbesgue INtegrAL
Radio Progs - M331 Integration and Normal Spaces
Prog 1 (U.1) Completeness
Prog 2 (U.2) The Integral of a Step Function

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Prog 5 (U.8-9) Apps. of the Convergence Theorems
Prog 6 (U.10) Lebesgue Measure

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