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Unit 1 The Real Numbers

Positive real numbers
Throughout this course, a real number \( x \) will be called positive if \( x \geq 0 \) and strictly positive if \( x > 0 \).

The set \( \mathbb{Z}^+ \)
The set of natural numbers \( \{1, 2, 3, \ldots\} \) is denoted by \( \mathbb{Z}^+ \).

Basic properties of \( \mathbb{R} \)
1. The set \( \mathbb{R} \) of real numbers is assumed to have the familiar algebraic and order properties. More formally, \( \mathbb{R} \) is assumed to be an ordered field.

2. Axiom of Completeness
Let \( \{s_n\} \) be an increasing real sequence which is bounded above. Then \( \{s_n\} \) converges to a real number. That is, if

- for each \( n \in \mathbb{Z}^+ \), \( s_n \leq s_{n+1} \),

and there is a real number \( K \) such that

- for each \( n \in \mathbb{Z}^+ \), \( s_n \leq K \),

then there is a real number \( s \) such that, given any \( \varepsilon > 0 \), there is an \( N \) in \( \mathbb{Z}^+ \) such that

\[ |s_n - s| < \varepsilon. \]

[This axiom implies that if \( \{t_n\} \) is a decreasing real sequence which is bounded below, so that

- for each \( n \in \mathbb{Z}^+ \), \( t_n \geq t_{n+1} \),

and there is a real number \( K \) such that

- for each \( n \in \mathbb{Z}^+ \), \( t_n \geq K \),

then \( \{t_n\} \) converges to a real number.]

Archimedean Property
There is no real number \( K \) such that

\[ \text{for all } n \in \mathbb{Z}^+, \quad n < K. \]

[This can be deduced from the Axiom of Completeness.]

Intervals
Let \( a, b \in \mathbb{R} \) and \( a \leq b \). Then the closed interval \([a, b]\), the open interval \((a, b)\) and the half-open intervals \([a, b)\) and \((a, b]\) are defined by

- \([a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}\),
- \((a, b) = \{x \in \mathbb{R} : a < x < b\}\),
- \([a, b) = \{x \in \mathbb{R} : a \leq x < b\}\),
- \((a, b] = \{x \in \mathbb{R} : a < x \leq b\}\).

These sets are called bounded intervals of \( \mathbb{R} \), each of length \( b - a \).

The sets

- \([a, \infty) = \{x \in \mathbb{R} : x \geq a\}\),
- \((a, \infty), (-\infty, a], (-\infty, a), \text{ defined in an analogous way, and}
- \((-\infty, \infty) = \mathbb{R}, \text{ are called unbounded intervals of } \mathbb{R}.\)
Convergent sequences
1. If the real sequence \( \{s_n\} \) converges to the limit \( s \), we often write \( s_n \to s \) or \( \lim s_n = s \).

2. Let \( \{s_n\} \) and \( \{t_n\} \) be real sequences which converge to \( s \) and \( t \) respectively. Then
   (a) \( s_n + t_n \to s + t \);
   (b) \( s_n - t_n \to s - t \);
   (c) \( s_n t_n \to st \);
   (d) if \( t_n \neq 0 \) for each \( n \) and \( t \neq 0 \), \( s_n/t_n \to s/t \);
   (e) if \( s_n \geq t_n \) for each \( n \), \( s \geq t \).

Convergent and absolutely convergent series
1. Let \( \{a_n\} \) be a real sequence. Then the series \( \sum a_n \) is said to converge if the sequence \( \{s_n\} \) of partial sums
   \[ s_n = a_1 + a_2 + \cdots + a_n \]
   converges. If \( \{s_n\} \) converges to \( s \), then \( s \) is called the sum of the series and this number is denoted by
   \[ \sum a_n \quad \text{or} \quad a_1 + a_2 + a_3 + \cdots \]
   The series \( \sum a_n \) is said to be absolutely convergent if the series \( \sum |a_n| \) is convergent.

2. The series \( \sum a_n \) of real numbers is absolutely convergent if and only if there are real sequences \( \{b_n\}, \{c_n\} \) such that each of the series \( \sum b_n, \sum c_n \) is convergent and, for each \( n \), \( b_n, c_n \geq 0 \) and \( a_n = b_n - c_n \).

[This result implies that every absolutely convergent series of real numbers is convergent.

To prove this result, the sequences \( \{a^+_n\} \) and \( \{a^-_n\} \) defined by
\[
\begin{align*}
a^+_n &= \begin{cases} a_n, & \text{if } a_n \geq 0, \\ 0, & \text{if } a_n < 0, \end{cases} \\
a^-_n &= \begin{cases} -a_n, & \text{if } a_n \leq 0, \\ 0, & \text{if } a_n > 0, \end{cases}
\end{align*}
\]
are introduced, where \( a^+_n \) and \( a^-_n \) are positive, \( a_n = a^+_n - a^-_n \) and \( |a_n| = a^+_n + a^-_n \).
Least Upper Bound Principle

A non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound.

[This Principle and the Axiom of Completeness are, in fact, equivalent statements, i.e. each of them can be derived from the other.]

Supremum, infimum

If $X$ is a non-empty subset of $\mathbb{R}$, which is bounded above, the least upper bound of $X$ is often referred to as the supremum of $X$, and denoted by $\text{sup} \ X$. If $X$ is bounded below, then $X$ has a greatest lower bound, which is often referred to as the infimum of $X$, and denoted by $\text{inf} \ X$.  

Countable set

1. A subset $S$ of $\mathbb{R}$ is said to be countable if its elements can be arranged as the terms of a sequence.

2. The set $\mathbb{Q}$ of rational numbers is countable.

3. The union of a sequence of countable sets is countable.

4. $\mathbb{R}$ is not countable.

Null set

1. A subset $S$ of $\mathbb{R}$ is called a null set if, for any $\varepsilon > 0$, there is a sequence $\{(a_n, b_n)\}$ of open intervals such that

$$ S \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty}(b_n - a_n) \leq \varepsilon. $$

2. A countable subset of $\mathbb{R}$ is null.

3. The union of a sequence of null sets is again a null set.

4. Cantor's ternary set is an example of a null set which is not countable.

Covering

Let $U$ be a collection of open intervals covering the closed interval $[a, b]$, in the sense that every element of $[a, b]$ belongs to at least one of the sets in $U$. Then there are a finite number of sets in $U$ which cover $[a, b]$.

[This result is a special case of the Heine-Borel Theorem.]

Functions of two variables

1. The set $\mathbb{R}^2$ consists of all pairs $(x_1, x_2)$ of real numbers. We write $x = (x_1, x_2)$ and call $x$ a point of $\mathbb{R}^2$. The distance between points $x$ and $y$ in $\mathbb{R}^2$ is

$$ |x - y|_2 = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^\frac{1}{2}. $$

[The subscript 2 distinguishes this from the absolute value.]

2. Let $D \subseteq \mathbb{R}^2$ be a subset of $\mathbb{R}^2$ and $x$ a point of $D$. Let $F : D \rightarrow \mathbb{R}$ be a function defined on $D$. $F$ has the limit $\ell$ at $x$ if for any $\varepsilon > 0$ we can find a $\delta > 0$, depending on $\varepsilon$ and $x$, such that

$$ |F(y) - \ell| < \varepsilon \quad \text{whenever} \quad |x - y|_2 < \delta. $$

If this is so, we write

$$ \lim_{y \to x} F(y) = \ell. $$
3. If
\[ \lim_{{y \to x}} F(y) = F(x), \]
we say that \( F \) is continuous at \( x \in D \). If \( F \) is continuous at all points of \( D \in \mathbb{R}^2 \), we say that \( F \) is continuous on \( D \).

4. Many more results are true for functions of two variables. Two in particular will be of use to us in this course. We omit proofs, which are, in any event, not very difficult as proofs go.

(a) If \( I \) and \( J \) are intervals in \( \mathbb{R} \), and if \( f : I \to \mathbb{R} \) and \( g : J \to \mathbb{R} \) are continuous functions, then
\[ F(x, y) = f(x)g(y), \quad x \in I, \ y \in J, \]
defines a continuous function \( F : I \times J \to \mathbb{R} \).

(b) If \( K \) is an interval in \( \mathbb{R}^2 \), and if \( F, G : K \to \mathbb{R} \) are continuous functions, then for any real numbers \( a \) and \( b \), the function
\[ H(x, y) = af(x, y) + bg(x, y) \]
is continuous on \( K \). Note that this includes the special cases when either \( a \) or \( b \) is zero.

**Unit 2  The Riemann Integral**

This unit is not examinable.

**Partition, mesh**

A **partition** of an interval \([a, b]\) is a finite set
\[ P = \{x_0, x_1, \ldots, x_n\} \]
of points of \([a, b]\), where
\[ a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b. \]
The **mesh** of such a partition \( P \), \( ||P|| \), is defined by
\[ ||P|| = \max_{1 \leq i \leq n} (x_i - x_{i-1}). \]

\( P \) is called a **standard partition** of \([a, b]\) if each of the intervals \([x_{i-1}, x_i], \ i = 1, \ldots, n, \) is of the same length.

**Upper and lower Riemann sums**

Let \( f \) be a bounded function on an interval \([a, b]\).

(a) For each partition \( P = \{x_0, x_1, \ldots, x_n\} \) of \([a, b]\), the **upper** and **lower** Riemann sums of \( f \), corresponding to \( P \), \( U(f, P) \) and \( L(f, P) \), are defined by
\[ U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}), \quad L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \]
where, for \( i = 1, \ldots, n, \)
\[ M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}, \quad m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}. \]

(b) The set \( \{U(f, P) : P \) a partition of \([a, b]\)\} of all upper Riemann sums of \( f \) is bounded. So is the set \( \{L(f, P) : P \) a partition of \([a, b]\)\}. App
The upper and lower (Riemann) integrals of \( f \), \( \int_a^b f \) and \( \int_a^b f^* \), are defined by

\[
\int_a^b f = \inf \{ U(f, P) : P \text{ a partition of } [a, b] \},
\]
\[
\int_a^b f^* = \sup \{ L(f, P) : P \text{ a partition of } [a, b] \}.
\]

The collection \( BR[a, b] \), the Riemann integral on \( [a, b] \)

It can be shown that \( \int_a^b f \geq \int_a^b f^* \) and that the inequality can be strict. \( f \) is said to be (Riemann) integrable on \( [a, b] \) if

\[
\int_a^b f = \int_a^b f^*;
\]

in this case, the (Riemann) integral of \( f \) on \( [a, b] \), \( \int_a^b f \), is defined by

\[
\int_a^b f = \int_a^b f^* = \int_a^b f^*.
\]

The collection of all bounded functions on \( [a, b] \), which are integrable on \( [a, b] \) in this sense, is denoted by \( BR[a, b] \).

Refinement

If \( P \) and \( Q \) are partitions of \( [a, b] \) and \( P \) is a subset of \( Q \), then \( Q \) is called a refinement of \( P \).

Definition T.2.1(2), page 8

Convergence in refinement

If \( f \) is a bounded function on \( [a, b] \), the upper Riemann sums \( U(f, P) \) are said to converge to the number \( U(f) \) in the sense of refinement if, given any \( \varepsilon > 0 \), there is a partition \( P(\varepsilon) \) such that

\[
\text{for every refinement} \ P \ \text{of} \ P(\varepsilon), \ |U(f, P) - U(f)| < \varepsilon.
\]

There is a similar definition of the convergence of the lower sums \( L(f, P) \) in the sense of refinement.

Definition T.2.1(2), page 8

Convergence in mesh

1. If \( f \) is a bounded function on \( [a, b] \), the upper Riemann sums \( U(f, P) \) are said to converge to the limit \( U'(f) \) in the sense of mesh if, given any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that

\[
\text{for any partition} \ P \ \text{of} \ [a, b] \ \text{with} \ ||P|| < \delta, \ |U(f, P) - U'(f)| < \varepsilon.
\]

There is a similar definition of the convergence of the \( L(f, P) \) in the sense of mesh.

Definition T.2.1(2), page 8

2. If \( f \in BR[a, b] \), then the upper and lower Riemann sums of \( f \) converge in the sense of refinement and in the sense of mesh to \( \int_a^b f \): in particular

\[
\lim_{n \to \infty} U(f, P_n) = \int_a^b f = \lim_{n \to \infty} L(f, P_n)
\]

where \( P_n \) is the standard partition of \( [a, b] \) with \( ||P_n|| = (b - a)/n \).

Conversely, if \( f \) is a bounded function on \( [a, b] \) and the upper and lower Riemann sums of \( f \) converge in the sense of refinement or in the sense of mesh to the same number, then \( f \in BR[a, b] \).

Theorem T.2.1, page 8
Integrable functions

1. **Riemann’s Criterion**

   Let \( f \) be a bounded function on \([a, b]\). Then \( f \in BR[a, b] \) if and only if, given any \( \epsilon > 0 \), there is a partition \( P \) of \([a, b]\), depending on \( \epsilon \) in general, such that
   \[
   U(f, P) - L(f, P) < \epsilon.
   \]

2. **If \( f \) is continuous on \([a, b]\), then \( f \in BR[a, b] \).**

   The conclusion still holds if \( f \) is continuous at all but a finite number of points of \([a, b]\).

3. **(a) \( BR[a, b] \) is a real vector space, that is, if \( f, g \in BR[a, b] \) and \( c, d \in \mathbb{R} \), then \( cf + dg \in BR[a, b] \).**

   **(b) If \( f, g \in BR[a, b] \), then \( fg \in BR[a, b] \); further, if there is a \( \delta > 0 \) such that, for each \( x \in [a, b], \ |g(x)| \geq \delta \), then \( f/g \in BR[a, b] \).**

   **(c) If \( f \in BR[a, b] \) and \([c, d] \subset [a, b]\), then the function \( g \) defined on \([c, d]\) by \( g(x) = f(x), \ x \in [c, d]\), belongs to \( BR[c, d]\).**

   **(d) If \( f \in BR[a, b] \), then \(|f| \in BR[a, b]|\).**

Properties of the integral

1. **(a) The Riemann integral on \([a, b]\) is a linear transformation of \( BR[a, b] \) into \( \mathbb{R} \), that is, if \( f, g \in BR[a, b] \) and \( c, d \in \mathbb{R} \), then**

   \[
   \int_a^b (cf + dg) = c \int_a^b f + d \int_a^b g.
   \]

   **(b) If \( x, y, z \) are any three numbers in any order in \([a, b]\), then, for any \( f \in BR[a, b] \),**

   \[
   \int_x^y f + \int_y^z f = \int_x^z f.
   \]

   **(c) The Riemann integral on \([a, b]\) is order-preserving, that is, if \( f, g \in BR[a, b] \) and \( f \geq g \) in the sense that for each \( x \in [a, b], \ f(x) \geq g(x) \), then**

   \[
   \int_a^b f \geq \int_a^b g.
   \]

2. **The Triangle Inequality for Integrals**

   For any \( f \in BR[a, b] \),

   \[
   \left| \int_a^b f \right| \leq \int_a^b |f|.
   \]

3. **For any two functions \( f, g \in BR[a, b] \),**

   \[
   \left( \int_a^b fg \right)^2 \leq \int_a^b f^2 \int_a^b g^2.
   \]

4. **The Second Fundamental Theorem of Calculus**

   If \( f : [a, b] \rightarrow \mathbb{R} \) has a derivative \( f' \) which is Riemann integrable on \([a, b]\), then

   \[
   \int_a^b f' = f(b) - f(a).
   \]
5. The First Fundamental Theorem of Calculus

If \( f \) is continuous on \([a, b]\) and the function \( F \) is defined on \([a, b]\) by

\[
F(x) = \int_a^x f,
\]

then \( F \) is differentiable and \( F' = f \).

Integration techniques

1. Integration by Parts

If \( f, g : [a, b] \to \mathbb{R} \) have derivatives \( f', g' \in BR[a, b] \), then

\[
\int_a^b f'g = [fg]_a^b - \int_a^b fg',
\]

where \([fg]_a^b\) denotes \( f(b)g(b) - f(a)g(a) \).

2. Integration by Substitution

If \( f : [a, b] \to \mathbb{R} \) has a derivative \( f' \in BR[a, b] \),

\( g : [\alpha, \beta] \to \mathbb{R} \) is one-one, has derivative \( g' \in BR[\alpha, \beta] \),

\( g(\alpha) = a, g(\beta) = b \) and \( (f \circ g)' g' \in BR[\alpha, \beta] \), then

\[
\int_a^b f' = \int_{\alpha}^{\beta} (f' \circ g) g'.
\]

In Leibniz notation, this reads

\[
\int_a^b f'(x) dx = \int_{\alpha}^{\beta} f'(g(t))g'(t) dt.
\]

Singular integrals: first kind

1. We say that the integral

\[
\int_a^\infty f
\]

converges if, for each \( b > a, f \in BR[a, b] \) and the limit

\[
\lim_{b \to \infty} \int_a^b f
\]

exists; in this case, we write

\[
\int_a^\infty f = \lim_{b \to \infty} \int_a^b f.
\]

In a similar way, we define

\[
\int_{-\infty}^b f = \lim_{a \to -\infty} \int_a^b f
\]

when it exists. We define

\[
\int_{-\infty}^\infty f = \lim_{b \to \infty} \int_0^b f + \lim_{a \to -\infty} \int_a^0 f = \int_0^\infty f + \int_{-\infty}^0 f,
\]

only if both \( \int_0^\infty f \) and \( \int_{-\infty}^0 f \) converge.

Any of these integrals is called a singular integral of the first kind.
2. **Comparison Test**

If \( 0 \leq f(x) \leq g(x) \) for all \( x \geq a, f \in BR[a, b] \) for all \( b > a \) and

\[
\int_a^\infty g
\]

converges, then

\[
\int_a^\infty f
\]

converges.

**Singular integrals: second kind**

1. If \( f \in BR[a + \varepsilon, b] \) for each \( \varepsilon \in (0, b - a) \), but \( f \) is not necessarily bounded on \([a, b]\), we say that

\[
\int_a^b f
\]

converges if the limit

\[
\lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f
\]

exists: in this case, we write

\[
\int_a^b f = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f,
\]

such an integral is called a *singular integral of the second kind*.

2. **Comparison Test**

If \( 0 \leq f(x) \leq g(x) \) for all \( x \in [a, b], f \in BR[a + \varepsilon, b] \) for all \( \varepsilon \in (0, b - a) \) and

\[
\int_a^b g = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b g
\]

converges as a singular integral of the second kind, then

\[
\int_a^b f = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f
\]

converges as a singular integral of the second kind and

\[
\int_a^b f \leq \int_a^b g.
\]

**Absolute convergence**

1. For any function \( f : [a, \infty) \to \mathbb{R} \) such that \( f \in BR[a, b] \) for all \( b > a \), we say that the singular integral \( \int_a^\infty f \) is *absolutely convergent* whenever the integral \( \int_a^\infty |f| \) is convergent.

2. Let \( f \in BR[a, b] \) for all \( b > a \). If the singular integral \( \int_a^\infty f \) is absolutely convergent, then it is convergent.

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*Proposition T.2.3.1, page 28*

*Definition T.2.3.2, page 30*

*Proposition T.2.3.2, page 31*

*Definition T.2.3.3, page 32*

*Proposition T.2.3.3, page 32*
Laplace transforms: exponential order

1. Let \( f : [0, \infty) \rightarrow \mathbb{R} \).
   (a) The Laplace transform of \( f \) is the function \( F \) defined by
   \[
   F(x) = \int_0^\infty e^{-xt} f(t) \, dt
   \]
   for all \( x \) for which the integral converges: if there is no such \( x \), then we say that \( f \) has no Laplace transform.
   (b) \( f \) is said to have exponential order with parameter \( \alpha \) if there is a constant \( M \) such that
   \[
   |f(t)| \leq Me^{\alpha t}.
   \]
   When \( \alpha = 1 \), we often abbreviate this to ‘\( f \) is of exponential order’.

2. Existence Theorem
   If \( f : [0, \infty) \rightarrow \mathbb{R} \) has exponential order with parameter \( \alpha \) and \( f \in B^2[0,T] \) for all \( T > 0 \), then the integral
   \[
   \int_0^\infty e^{-xt} f(t) \, dt
   \]
   is absolutely convergent for all \( x > \alpha \), and so the Laplace transform of \( f \) is defined, at least, on \( (\alpha, \infty) \).

Double Integrals

1. A partition of a rectangle \([a, b] \times [c, d]\) is an ordered pair \((P', P'')\), where \( P' \) is a partition of \([a, b]\) and \( P'' \) is a partition of \([c, d]\).

2. Let \( F \) be a bounded function on a rectangle \( R = [a, b] \times [c, d] \).
   (a) For each partition \( Q = (P', P'') \) of \( R \), say
   \[
   P' = \{x_0, x_1, \ldots, x_m\} \quad \text{and} \quad P'' = \{y_0, y_1, \ldots, y_n\},
   \]
   define
   \[
   U(F, Q) = \sum_{i=1}^m \sum_{j=1}^n M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}),
   \]
   \[
   L(F, Q) = \sum_{i=1}^m \sum_{j=1}^n m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}),
   \]
   where, for \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \),
   \[
   M_{ij} = \sup \{F(x, y) : x \in [x_{i-1}, x_i] \text{ and } y \in [y_{j-1}, y_j]\},
   \]
   \[
   m_{ij} = \inf \{F(x, y) : x \in [x_{i-1}, x_i] \text{ and } y \in [y_{j-1}, y_j]\};
   \]
   \( U(F, Q) \) and \( L(F, Q) \) are called respectively the upper and lower Riemann sums of \( F \), corresponding to the partition \( Q \).
   (b) Write
   \[
   \iint_R F = \inf \{U(F, Q) : Q \text{ a partition of } R\},
   \]
   \[
   \iint_R F = \sup \{L(f, Q) : Q \text{ a partition of } R\}.
   \]
   \( \iint_R F \) and \( \iint_R F \) are called respectively the upper and lower Riemann integrals of \( F \) over \( R \).
   (c) \( F \) is said to be Riemann integrable over \( R \) if
   \[
   \iint_R F = \iint_R F,
   \]
   in which case, the Riemann integral of \( F \) over \( R \), \( \iint_R F \), is defined to be this common value.
3. Extensions of the above definition

(a) Let $\mathcal{D}$ be a bounded non-rectangular region in $\mathbb{R}^2$ and $F: \mathcal{D} \rightarrow \mathbb{R}$ be a bounded function. Let $\mathcal{R}$ be a rectangle in $\mathbb{R}^2$, containing $\mathcal{D}$, and define $G$ on $\mathcal{R}$ by

$$
G(x, y) = \begin{cases} 
F(x, y) & \text{if } (x, y) \in \mathcal{D}, \\
0 & \text{if } (x, y) \in \mathcal{R} \setminus \mathcal{D}.
\end{cases}
$$

Then $F$ is said to be Riemann integrable over $\mathcal{D}$ if $G \in BR(\mathcal{R})$ and, in this case, $\int_{\mathcal{D}} F$, the Riemann integral of $F$ over $\mathcal{D}$, is defined by

$$
\int_{\mathcal{D}} F = \int_{\mathcal{R}} G.
$$

(b) If $\mathcal{D}$ is an unbounded region in $\mathbb{R}^2$ or the restriction that $F: \mathcal{D} \rightarrow \mathbb{R}$ is bounded is dropped, then we could try to define $\int_{\mathcal{D}} F$ as a limit, as in the earlier discussion of singular integrals.

Evaluation of double integrals

1. If $F$ is integrable on $\mathcal{R} = [a, b] \times [c, d]$ and, for each $y \in [c, d], \int_a^b F(x, y) \, dx$ exists, that is the function $x \mapsto F(x, y)$ belongs to $BR[a, b]$, then the repeated integral

$$
\int_a^b \left( \int_a^b F(x, y) \, dx \right) \, dy
$$

exists, and equals the Riemann integral of $F$ over $\mathcal{R}$:

$$
\int_{\mathcal{R}} F = \int_c^d \left( \int_a^b F(x, y) \, dx \right) \, dy.
$$

[It is common practice to omit the large brackets in repeated integral expressions, so that the equation above becomes

$$
\int_{\mathcal{R}} F = \int_c^d \int_a^b F(x, y) \, dx \, dy.
$$

2. Fubini's Theorem

If, in addition to the conditions of the above result, it is assumed that $\int_a^b F(x, y) \, dy$ exists for each $x \in [a, b]$, then

$$
\int_{\mathcal{R}} F = \int_c^d \int_a^b F(x, y) \, dx \, dy = \int_a^b \int_c^d F(x, y) \, dy \, dx.
$$

[If, for example, $F$ is continuous on $\mathcal{R}$, then the conditions of Fubini's Theorem are satisfied.]
3. **Two-dimensional Substitution Rule**

A change of variables in a two-dimensional integral

\[
\int \int_D f(x, y) \, dx \, dy
\]

involves replacing \(x, y\) by two functions \(x(t, u), y(t, u)\) of two variables \(t, u\), subject to appropriate conditions. The analogue of \(dx/dt\) in one variable is the so-called Jacobian determinant

\[
\begin{vmatrix}
\frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\
\frac{\partial y}{\partial t} & \frac{\partial y}{\partial u}
\end{vmatrix}
\]

The formula is

\[
\int \int_D f(x, y) \, dx \, dy = \int \int_S f(x(t, u), y(t, u)) \left| \frac{\partial(x, y)}{\partial(t, u)} \right| \, dt \, du,
\]

where \(S\) is the region of integration expressed in terms of the new variables \(t, u\).

---

**Unit 3  Step Functions**

### Characteristic functions

1. Let \(S\) be a subset of \(\mathbb{R}\). Then the *characteristic function* of \(S\) is the function \(\chi_S\) defined by

\[
\chi_S(x) = \begin{cases} 
1, & \text{if } x \in S, \\
0, & \text{if } x \notin \mathbb{R} \setminus S.
\end{cases}
\]

2. Denote by \(C\) the set of characteristic functions of bounded intervals in \(\mathbb{R}\). If \(\phi \in C\), say \(\phi = \chi_I\), define

\[
\int \phi = \ell(I),
\]

the length of \(I\).

[If \(I\) has end points \(a, b\) and \(a \leq b\), then \(\ell(I) = b - a\).]

### Step functions

1. Denote by \(S\) the set of all linear combinations of functions in \(C\), i.e. the set of all functions \(\phi\) of the form

\[
\phi = c_1 \chi_{I_1} + c_2 \chi_{I_2} + \cdots + c_r \chi_{I_r},
\]

where \(r \in \mathbb{Z}^+\), \(I_1, I_2, \ldots, I_r\) are bounded intervals in \(\mathbb{R}\) and \(c_1, c_2, \ldots, c_r\) are real numbers. \(S\) is simply the real vector space of functions spanned by \(C\). The functions in \(S\) are called *step functions* (on \(\mathbb{R}\)).

[For example, \(\phi = \frac{3}{2} \chi_{[1,3]} - 2 \chi_{[2,3)}\) belongs to \(S\). The graph of \(\phi\) is as shown.]
Note that $\phi = \frac{3}{2} \chi_{[1,2]} - \frac{1}{2} \chi_{[2,3]} + \frac{3}{2} \chi_{[3,4]}$ gives a second description of $\phi$ as a linear combination of functions in $C$, this time as a linear combination of characteristic functions of disjoint bounded intervals.

2. Any step function can be expressed as a linear combination of characteristic functions of disjoint bounded intervals. [Proposition 3.1.1, W page 25]

### Integration of step functions

1. If $\phi \in S$, say $\phi = \sum_{j=1}^{r} c_j \chi_{I_j}$ where $I_1, I_2, \ldots, I_r$ are bounded intervals and $c_1, c_2, \ldots, c_r$ are real numbers, it seems natural to define

$$\int \phi = \sum_{j=1}^{r} c_j \ell(I_j).$$

It must, however, first be shown that the value of $\int \phi$ obtained is independent of the description of $\phi$ as a linear combination of functions in $C$.

2. If $I_1, I_2, \ldots, I_r, J_1, J_2, \ldots, J_s$ are bounded intervals, $c_1, c_2, \ldots, c_r, d_1, d_2, \ldots, d_s$ are real numbers and

$$\sum_{j=1}^{r} c_j \chi_{I_j} = \sum_{k=1}^{s} d_k \chi_{J_k},$$

then

$$\sum_{j=1}^{r} c_j \ell(I_j) = \sum_{k=1}^{s} d_k \ell(J_k).$$

3. If $\phi \in S$, say $\phi = \sum_{j=1}^{r} c_j \chi_{I_j}$ where $I_1, I_2, \ldots, I_r$ are bounded intervals and $c_1, c_2, \ldots, c_r$ are real numbers, define

$$\int \phi = \sum_{j=1}^{r} c_j \ell(I_j).$$

[The above result shows that this definition is unambiguous.]

### Properties

1. (a) $\int$ is a linear transformation of $S$ into $\mathbb{R}$, i.e. for all $\phi, \psi \in S$ and $a, b \in \mathbb{R}$,

$$\int (a\phi + b\psi) = a \int \phi + b \int \psi.$$  

[Weir refers to $\int$ as a linear operator on $S$, though this conflicts with standard mathematical terminology. A linear transformation of a real vector space $V$ into $\mathbb{R}$ is usually referred to as a linear functional on $V$.]

(b) $\int$ is order-preserving, i.e. if $\phi, \psi \in S$ and $\phi \geq \psi$, meaning $\phi(x) \geq \psi(x)$ for each $x$ in $\mathbb{R}$, then

$$\int \phi \geq \int \psi.$$  

[Proposition 3.1.3, W page 28]

2. Let $\phi, \psi \in S$. Then

(a) each of the functions $|\phi|, \max\{\phi, \psi\}, \min\{\phi, \psi\}$ belongs to $S$;

(b) $|\int \phi| \leq \int |\phi|$. 


The functions \( \max\{\phi, \psi\} \) and \( \min\{\phi, \psi\} \) are defined on \( \mathbb{R} \) in the obvious way, namely, for each \( x \in \mathbb{R} \),
\[
\max\{\phi, \psi\}(x) = \max\{\phi(x), \psi(x)\}
\]
and
\[
\min\{\phi, \psi\}(x) = \min\{\phi(x), \psi(x)\},
\]
where, for any \( a, b \in \mathbb{R} \),
\[
\max\{a, b\} = \begin{cases} 
  a, & \text{if } a \geq b, \\
  b, & \text{if } a < b,
\end{cases}
\]
and
\[
\min\{a, b\} = \begin{cases} 
  b, & \text{if } a \geq b, \\
  a, & \text{if } a < b.
\end{cases}
\]
It is easily checked that
\[
\max\{\phi, \psi\} = \frac{1}{2}(\phi + \psi + |\phi - \psi|),
\]
\[
\min\{\phi, \psi\} = \frac{1}{2}(\phi + \psi - |\phi - \psi|)
\]
and
\[
|\phi - \psi| = \max\{\phi, \psi\} - \min\{\phi, \psi\}.
\]
These definitions and relations hold, in fact, for any real-valued functions with common domain.

**Unit 4 The Lebesgue Integral on \( \mathbb{R} \)**

Almost everywhere (almost all)

If a property holds for all real numbers \( x \) outside some null set, the property is said to hold *almost everywhere* or to hold for *almost all* \( x \).

**Convergence and null sets**

1. If \( \{\phi_n\} \) is an increasing sequence of step functions for which \( \{\int \phi_n\} \) converges, then \( \{\phi_n\} \) converges almost everywhere, i.e. there is a null subset \( N \) of \( \mathbb{R} \) such that,

   for each \( x \in \mathbb{R} \setminus N \), \( \{\phi_n(x)\} \) converges.

2. If \( S \) is a null set in \( \mathbb{R} \), then there is an increasing sequence \( \{\psi_n\} \) of step functions for which the sequence \( \{\int \psi_n\} \) converges and \( \{\psi_n(x)\} \) diverges for each \( x \) in \( S \).

   [This result could be used as a definition of null sets, and some authors do just that.]

**The space \( L^{inc} \)**

Denote by \( L^{inc} \) the set of all functions \( f \) on \( \mathbb{R} \) for which there is an increasing sequence \( \{\phi_n\} \) of step functions such that
\[
\{\phi_n\} \text{ converges to } f \text{ almost everywhere}
\]
and
\[
\{\int \phi_n\} \text{ converges.}
\]
Integration of functions in $L^\text{inc}$

1. If $f \in L^\text{inc}$ and $\{\phi_n\}$ is an increasing sequence of step functions satisfying the above conditions, we would like to define

$$\int f = \lim \int \phi_n.$$

We first show that such a definition is unambiguous.

2. Let $\{\phi_n\}$ be a decreasing sequence of positive step functions converging almost everywhere to 0. Then

$$\lim \int \phi_n = 0.$$

3. Suppose that the functions $f$, $g$ of $L^\text{inc}$ are determined almost everywhere by increasing sequences $\{\phi_n\}$, $\{\psi_n\}$ of step functions respectively, and that $f \geq g$ almost everywhere. Then

$$\lim \int \phi_n \geq \lim \int \psi_n.$$

4. If $f \in L^\text{inc}$, define

$$\int f = \lim \int \phi_n,$$

where $\{\phi_n\}$ is an increasing sequence of step functions such that $\{\phi_n\}$ converges to $f$ almost everywhere and $\{\int \phi_n\}$ converges.

[The above result implies that this definition is unambiguous. Also $L^\text{inc} \supset S$, and $\int$, defined on $L^\text{inc}$, is an extension of the integral defined in Unit 3.]

The space $L^1$

1. Denote by $L^1$ the set of all functions $f$ of the form

$$f = g - h,$$

where $g, h \in L^\text{inc}$.

If $f \in L^1$, define

$$\int f = \int g - \int h,$$

where $g, h \in L^\text{inc}$ and $f = g - h$.

$L^1$ is called the space of (Lebesgue) integrable functions and the function $\int$ on $L^1$ is called the Lebesgue integral.

[Again the definition of $\int$ is unambiguous. It is an extension of the integral on $L^\text{inc}$ defined above.]
Unit 5  Definite and Indefinite Integrals

Integrals on intervals

1. Let $I$ be an interval in $\mathbb{R}$. Denote by $L^1(I)$ the set of all functions $f : \mathbb{R} \to \mathbb{R}$ for which $f_x \in L^1$. If $f \in L^1(I)$, $f$ is said to be (Lebesgue) integrable on $I$ and we write

$$\int_I f = \int f_x :$$

$\int_I f$ is called the (Lebesgue) integral of $f$ on $I$. If $I$ is bounded and has end points $a$ and $b$ with $a \leq b$, then $\int_I f$ is denoted by

$$\int_a^b f,$$

or, in the classical notation,

$$\int_a^b f(x) \, dx.$$

Similar notations are used when $I$ is unbounded.

2. Let $a, b, c \in \mathbb{R}$ and $a < b < c$. If $f \in L^1[a, b]$ and $f \in L^1[b, c]$, then

$$\int_a^c f = \int_a^b f + \int_b^c f.$$  

3. The Mean Value Theorem for Integrals

If $f \in L^1[a, b]$ and there are real numbers $m, M$ such that

for each $x \in [a, b], \quad m \leq f(x) \leq M,$

then

$$m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a).$$

Sufficient conditions for integrability

1. Let $f$ be a function which vanishes outside the interval $[a, b]$. If $f$ is bounded and the points of discontinuity of $f$ form a null set, then $f \in L^1$.

2. If $f$ is bounded on $[a, b]$ and the points of discontinuity of $f$ on $[a, b]$ form a null set, then $f \in L^1[a, b]$.

[It can be shown that a bounded function $f$ on $[a, b]$ is Riemann integrable on $[a, b]$, as defined in Unit 2, if and only if the points of discontinuity of $f$ form a null set. Hence $BR[a, b] \subseteq L^1[a, b]$. Further, if $f \in BR[a, b]$, then the Riemann and Lebesgue integrals of $f$ on $[a, b]$ are equal.]

3. If either

(a) $f$ is continuous on $[a, b]$ or
(b) $f$ is monotonic on $[a, b],$

then $f$ is integrable on $[a, b]$.

Integration techniques

1. The Fundamental Theorem of Calculus

If the function $F$ has a continuous derivative $f$ on the closed interval $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$
2. Integration by Parts

If the functions $F, G$ have continuous derivatives $f, g$ respectively on $[a, b]$, then

$$\int_a^b Fg = \left[F G\right]_a^b - \int_a^b fG,$$

where $[F G]_a^b$ stands for $F(b)G(b) - F(a)G(a)$.

3. Integration by Substitution

Suppose that $G$ has a positive continuous derivative on the closed interval $[c, d]$, and write $a = G(c), b = G(d)$. Then, for any function $f$ which is continuous on the closed interval $[a, b]$,

$$\int_a^b f(x) \, dx = \int_c^d f(G(t)) G'(t) \, dt.$$ 

**Unit 6 The Lebesgue Integral on $\mathbb{R}^k$**

**Bounded intervals in $\mathbb{R}^k$**

A subset $I$ of $\mathbb{R}^k$ is called a **bounded interval** in $\mathbb{R}^k$ if, for $i = 1, \ldots, k$, there are real numbers $a_i, b_i$ with $a_i \leq b_i$ such that

$$I = \{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k: \text{ for } i = 1, \ldots, k, a_i < x_i < b_i\}$$

where, in each case, $<$ is either $\leq$ or $<$. The measure of such an interval $I$, $m(I)$, is defined by

$$m(I) = (b_1 - a_1)(b_2 - a_2) \cdots (b_k - a_k).$$

[A bounded interval in $\mathbb{R}^k$ is just the Cartesian product $I_1 \times I_2 \times \cdots \times I_k$ of $k$ bounded intervals in $\mathbb{R}$. The measure of such an interval is the product of the lengths of the intervals $I_i$, $i = 1, \ldots, k$. Unbounded intervals in $\mathbb{R}^k$ can be defined in a similar way.]

**Characteristic functions**

Let $S$ be a subset of $\mathbb{R}^k$. Then the **characteristic function** of $S$ is the function $\chi_S$ defined by

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \in \mathbb{R}^k \setminus S. \end{cases}$$

**Step functions**

1. Denote by $\mathcal{S}$ the set of all functions $\phi$ of the form

$$\phi = c_1 \chi_{I_1} + c_2 \chi_{I_2} + \cdots + c_r \chi_{I_r},$$

where $r \in \mathbb{Z}^+$, $I_1, I_2, \ldots, I_r$ are bounded intervals in $\mathbb{R}^k$ and $c_1, c_2, \ldots, c_r$ are real numbers. $\mathcal{S}$ is a real vector space. The functions in $\mathcal{S}$ are called **step functions** (on $\mathbb{R}^k$).

2. Any step function can be expressed as a linear combination of characteristic functions of disjoint bounded intervals.
Integration of step functions

1. If \( I_1, I_2, \ldots, I_r, J_1, J_2, \ldots, J_s \) are bounded intervals, \( c_1, c_2, \ldots, c_r, d_1, d_2, \ldots, d_s \) are real numbers and
\[
\sum_{j=1}^{r} c_j \chi_{I_j} = \sum_{k=1}^{s} d_k \chi_{J_k},
\]
then
\[
\sum_{j=1}^{r} c_j m(I_j) = \sum_{k=1}^{s} d_k m(J_k).
\]

2. If \( \phi \in S \), say \( \phi = \sum_{j=1}^{r} c_j \chi_{I_j} \) where \( I_1, I_2, \ldots, I_r \) are bounded intervals and \( c_1, c_2, \ldots, c_r \) are real numbers, define
\[
\int \phi = \sum_{j=1}^{r} c_j m(I_j).
\]

[The above result shows that this definition is unambiguous.]

Properties

1. \( \int \) is an order-preserving linear operator (functional) on \( S \). \( \text{W pages 73, 75} \)
2. Let \( \phi, \psi \in S \). Then
   (a) each of the functions \( |\phi|, \max\{\phi, \psi\}, \min\{\phi, \psi\} \) belongs to \( S \); \( \text{Exercise 3, W page 76} \)
   (b) \( |\int \phi| \leq \int |\phi| \).

Null sets

1. A subset \( S \) of \( \mathbb{R}^k \) is called a null set if, for any \( \varepsilon > 0 \), there is a sequence \( \{I_n\} \) of bounded open intervals in \( \mathbb{R}^k \) such that
\[
S \subseteq \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad \sum_{n=1}^{\infty} m(I_n) < \varepsilon.
\]
2. Any countable set in \( \mathbb{R}^k \) is null. \( \text{Proposition 4.2.1, W page 77} \)
3. The union of a sequence of null sets in \( \mathbb{R}^k \) is itself a null set. \( \text{Proposition 4.2.2, W page 77} \)

Almost everywhere (almost all)

If a property holds for all points \( x \) of \( \mathbb{R}^k \) outside some null set, the property is said to hold almost everywhere or to hold for almost all \( x \).

Convergence and null sets

1. If \( \{\phi_n\} \) is an increasing sequence of step functions on \( \mathbb{R}^k \) for which the sequence \( \{\int \phi_n\} \) converges, then \( \{\phi_n\} \) converges almost everywhere, i.e. there is a null subset \( N \) of \( \mathbb{R}^k \) such that
   for each \( x \in \mathbb{R}^k \setminus N \), \( \{\phi_n(x)\} \) converges. \( \text{Theorem 4.2.1, W page 77} \)
2. If \( S \) is a null set in \( \mathbb{R}^k \), then there is an increasing sequence \( \{\psi_n\} \) of step functions for which the sequence \( \{\int \psi_n\} \) converges and \( \{\psi_n(x)\} \) diverges for each \( x \) in \( S \). \( \text{Converse, W page 77} \)
The space $L^{\text{inc}}(\mathbb{R}^k)$

Denote by $L^{\text{inc}}(\mathbb{R}^k)$ the set of functions $f$ on $\mathbb{R}^k$ for which there is an increasing sequence $\{\phi_n\}$ of step functions such that $\{\phi_n\}$ converges to $f$ almost everywhere and the sequence $\{\int \phi_n\}$ converges.

If $f \in L^{\text{inc}}(\mathbb{R}^k)$, define

$$\int f = \lim \int \phi_n,$$

where $\{\phi_n\}$ is an increasing sequence of step functions satisfying the above conditions.

(The proof that this definition is unambiguous is essentially the same as the proof of the corresponding result, discussed in Unit 4.)

The space $L^1(\mathbb{R}^k)$

1. Denote by $L^1(\mathbb{R}^k)$ the set of all functions $f$ of the form

   $$f = g - h,$$

   where $g, h \in L^{\text{inc}}(\mathbb{R}^k)$.

   If $f \in L^1(\mathbb{R}^k)$, define

   $$\int f = \int g - \int h,$$

   where $g, h \in L^{\text{inc}}(\mathbb{R}^k)$ and $f = g - h$.

   (This definition is unambiguous.)

   $L^1(\mathbb{R}^k)$, or simply $L^1$ if the space $\mathbb{R}^k$ is understood, is called the space of (Lebesgue) integrable functions on $\mathbb{R}^k$. The function $f$ on $L^1(\mathbb{R}^k)$ is called the Lebesgue integral. It is an extension of the integral on $S$, defined above. If $f \in L^1(\mathbb{R}^k)$, the classical notation for $\int f$ is

   $$\int f(x) \, dx$$

   or even

   $$\int f(x_1, \ldots, x_k) \, dx(x_1, \ldots, x_k).$$

2. (a) $L^1(\mathbb{R}^k)$ is a real vector space.

   (b) $\int$ is an order-preserving linear operator (functional) on $L^1(\mathbb{R}^k)$.  

3. Let $f, g \in L^1(\mathbb{R}^k)$. Then

   (a) each of the functions $|f|, \max\{f, g\}, \min\{f, g\}$ belongs to $L^1(\mathbb{R}^k)$;

   (b) $|\int f| \leq \int |f|$.

4. If $f \in L^1(\mathbb{R}^k)$ and $f_2 = f_1$ almost everywhere, then $f_2 \in L^1(\mathbb{R}^k)$ and $\int f_2 = \int f_1$.

Integrals on intervals

Let $I$ be an interval on $\mathbb{R}^k$ (not necessarily bounded). Denote by $L^1(I)$ the set of all functions $f : \mathbb{R}^k \to \mathbb{R}$ for which $f_{|I} \in L^1(\mathbb{R}^k)$. If $f \in L^1(I)$, $f$ is said to be (Lebesgue) integrable on $I$ and we write

$$\int_I f = \int f_{|I};$$

$\int_I f$ is called the (Lebesgue) integral of $f$ on $I$.  

Theorem 4.2.2, page 78  

Theorem 4.2.3, page 79
Sufficient conditions for integrability

1. Let \( f : \mathbb{R}^k \rightarrow \mathbb{R} \) be a function which vanishes outside a bounded interval \( I \) of \( \mathbb{R}^k \). If \( f \) is bounded and the points of discontinuity of \( f \) form a null set in \( \mathbb{R}^k \), then \( f \in L^1(\mathbb{R}^k) \).

2. Let \( I \) be a bounded interval in \( \mathbb{R}^k \). If \( f \) is bounded on \( I \) and the points of discontinuity of \( f \) on \( I \) form a null set, then \( f \in L^1(I) \).

3. If \( f \) is continuous on the bounded closed interval \( I \) in \( \mathbb{R}^k \), then \( f \) is integrable on \( I \).

Unit 7 Fubini's Theorem

Evaluating double integrals

1. Fubini's Theorem for \( \mathbb{R}^2 \)

If \( f \in L^1(\mathbb{R}^2) \), then

\[
\int \int f(x,y) \, dx \, dy = \left( \int f(x,y) \, dy \right) \, dx.
\]

(The above equation means that, for almost all \( x \) in \( \mathbb{R} \), the function \( f_x : y \mapsto f(x,y) \quad (y \in \mathbb{R}) \) belongs to \( L^1(\mathbb{R}) \), i.e. there is a null subset \( N \) of \( \mathbb{R} \) such that, for all \( x \in \mathbb{R} \setminus N \), \( f_x \in L^1(\mathbb{R}) \), the function \( F \) defined by

\[
F(x) = \begin{cases} 
\int f_x, & \text{if } x \in \mathbb{R} \setminus N, \\
0 \text{ (say),} & \text{otherwise,}
\end{cases}
\]

belongs to \( L^1(\mathbb{R}) \) and

\[
\int F = \int f.
\]

In the classical notation, this equation becomes

\[
\int F(x) \, dx = \left( \int f(x,y) \, dy \right) \, dx \\
= \int f \\
= \int f(x,y) \, dx \, dy.
\]

The result is proved in several steps. It is easily shown to be true when \( f \) is the characteristic function of a bounded interval in \( \mathbb{R}^2 \), and so, by linearity, when \( f \) is a step function on \( \mathbb{R}^2 \). The deduction that the result is true when \( f \in L^{inc}(\mathbb{R}^2) \) depends on the following.

If \( S \) is a null subset of \( \mathbb{R}^2 \), then, for almost all \( x \) in \( \mathbb{R} \),

\[
\{ y \in \mathbb{R} : (x,y) \in S \}
\]

is a null subset of \( \mathbb{R} \).

The final step, that the result is true when \( f \in L^1(\mathbb{R}^2) \), follows by linearity.
2. If $f \in L^1(\mathbb{R}^2)$, then

$$\int \left( \int f(x,y) \, dx \right) \, dy = \int \left( \int f(x,y) \, dy \right) \, dx.$$  

[This is, of course, an immediate consequence of 1. We note that the large brackets within the integrals are often omitted, and the result is expressed as

$$\int \int f(x,y) \, dx \, dy = \int \int f(x,y) \, dy \, dx.$$]

Evaluating multiple integrals

1. Fubini's Theorem for $\mathbb{R}^k$

If $f \in L^1(\mathbb{R}^k)$, where $k \geq 2$, $m$, $n$ are natural numbers such that $m + n = k$, and $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, then

$$\int f(x,y) \, dx \, dy = \int \left( \int f(x,y) \, dx \right) \, dy;$$

so, in particular,

$$\int f = \int \left( \int \left( \int f(x_1,x_2,\ldots,x_k) \, dx_k \right) \ldots dx_2 \right) \, dx_1.$$  

2. With the notation in the statement of the above result,

$$\int \left( \int f(x,y) \, dx \right) \, dy = \int \left( \int f(x,y) \, dy \right) \, dx;$$

it follows that the integral of $f$ can be evaluated in any order of the variables $x_1, x_2, \ldots, x_k$.

Measure, area, volume

If $S$ is a subset of $\mathbb{R}^k$ and $\chi_S \in L^1(\mathbb{R}^k)$, define the measure of $S$, $m(S)$, by

$$m(S) = \int \chi_S.$$

If $k = 2$, $m(S)$ is called the area of $S$; if $k = 3$, $m(S)$ is called the volume of $S$.

[This definition of measure extends the definition of the measure of a bounded interval in $\mathbb{R}^k$, given in Unit 6.]

Unit 8 The Monotone Convergence Theorem

Preliminary results

1. Let $\{f_n\}$ be an increasing sequence of functions in $L^\infty$ for which $\{\int f_n\}$ is bounded. Then $\{f_n\}$ converges almost everywhere to a function $f$ in $L^\infty$ and

$$\int f = \lim \int f_n.$$

[The material of Unit 6 shows that this result is true if the $f_n$ are step functions.]
2. Let \( f \in L^1 \). Then, given any \( \varepsilon > 0 \), there are \( g, h \) in \( L^\infty \) such that \( f = g - h \), where \( h \) is positive and
\[
\int h < \varepsilon.
\]
[If \( f \in L^1 \), then, by definition, there are \( g, h \) in \( L^\infty \) such that \( f = g - h \). This result shows that we may assume, in addition, that \( h \) is positive and \( \int h \) is small.]

**Monotone Convergence Theorem**

Let \( \{f_n\} \) be a monotone sequence of functions in \( L^1(\mathbb{R}^k) \) for which \( \{f_n\} \) is bounded. Then \( \{f_n\} \) converges almost everywhere to a function \( f \) in \( L^1(\mathbb{R}^k) \) and
\[
\int f = \lim \int f_n.
\]

**Corollary**

If \( f \) is a positive element of \( L^1 \) and \( \int f = 0 \), then \( f = 0 \) almost everywhere.

**Absolute Convergence Theorem**

Let \( \{a_n\} \) be a sequence of functions in \( L^1(\mathbb{R}^k) \) such that the series \( \sum_{j=1}^{\infty} \int |a_j| \) is convergent. Then there is a function \( f \) in \( L^1(\mathbb{R}^k) \) such that \( \sum_{j=1}^{\infty} a_j(x) \) converges to \( f(x) \) for almost all \( x \), and
\[
\int f = \sum_{j=1}^{\infty} \int a_j.
\]

**Integrability: unbounded functions, intervals**

Let \( \{I_n\} \) be an increasing sequence of intervals in \( \mathbb{R}^k \) whose union is the interval \( I \). If \( f \in L^1(I_n) \) for \( n \geq 1 \) and the integrals
\[
\int_{I_n} |f|
\]
are bounded above, then \( f \in L^1(I) \) and
\[
\int_I f = \lim \int_{I_n} f.
\]

**Techniques**

1. **Integration by Parts**

Let \( I \) be any interval in \( \mathbb{R} \). Suppose the functions \( F \) and \( G \) have continuous derivatives, \( f \) and \( g \) respectively, on \( I \), and \( Fg \) and \( fg \) are in \( L^1(I) \). Then, for any increasing sequence \( \{[a_n, b_n]\} \) of bounded, closed intervals having union \( I \),
\[
\lim_{n \to \infty} [FG]_{a_n}^{b_n} = \int_I Fg + \int_I fg.
\]

[Such a sequence \( \{[a_n, b_n]\} \) is sometimes referred to as a closed covering sequence for the interval \( I \). The result shows that \( \lim_{n \to \infty} [FG]_{a_n}^{b_n} \) is independent of the closed covering sequence for \( I \), and so may be denoted by \( [FG]_I \). The result can then be rearranged as
\[
\int_I Fg = [FG]_I - \int_I fg.
\]
2. **Integration by Substitution**

Let $I$ be any interval in $\mathbb{R}$. Suppose that $G$ is a real function with a positive continuous derivative on $I$, and $G(I)$ is the interval $J$. Let $f$ be a function which is continuous on $J$. Then the function $h$ defined on $I$ by

$$h(t) = f(G(t))G'(t)$$

is in $L^1(I)$ if and only if $f$ is in $L^1(J)$, in which case

$$\int_J f = \int_I h,$$

that is,

$$\int_J f(x) \, dx = \int_I f(G(t))G'(t) \, dt.$$

**The Gamma Function**

For each $\alpha > 0$,

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} \, dx = \lim_{n \to \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{\alpha-1} \, dx.$$

[The first equation is the definition of $\Gamma(\alpha)$.]

---

**Unit 9  The Dominated Convergence Theorem**

**Upper and lower limits**

1. Let $\{s_n\}$ be a bounded sequence of real numbers. For each $n \in \mathbb{Z}^+$, define $u_n = \sup\{s_n, s_{n+1}, \ldots\}$ and $\ell_n = \inf\{s_n, s_{n+1}, \ldots\}$. The sequence $\{u_n\}$ is decreasing and the sequence $\{\ell_n\}$ is increasing. Since each of these sequences is bounded, each of them is convergent. Their limits are called respectively the upper and lower limits of the sequence $\{s_n\}$, and are denoted by $\limsup s_n$ and $\liminf s_n$. Clearly $\limsup s_n \geq \liminf s_n$.

2. The bounded sequence $\{s_n\}$ of real numbers converges to the limit $s$ if and only if $\liminf s_n = \limsup s_n = s$.

**Dominated Convergence Theorem**

Let $\{f_n\}$ be a sequence of integrable functions which converges almost everywhere to a function $f$ and which is dominated by an integrable function $g$ in the sense that, for each $n \in \mathbb{Z}^+$,

$$|f_n| \leq g.$$

Then $f$ is integrable and

$$\int f = \lim \int f_n.$$
Corollary

The Bounded Convergence Theorem

Let $I$ be a bounded interval and $\{f_n\}$ be a sequence of functions in $L^1(I)$ which converges almost everywhere on $I$ to a function $f$. Suppose that there is a real number $K$ such that

for each $n \in \mathbb{Z}^+$ and each $x \in I$, $|f_n(x)| \leq K$.

Then $f \in L^1(I)$ and

$$\int_I f = \lim_{n \to \infty} \int_I f_n.$$ 

Fatou's Lemma

Let $\{f_n\}$ be a sequence of positive integrable functions which converges almost everywhere to a function $f$. Suppose that the sequence $\{f_n\}$ is bounded. Then $f$ is integrable and

$$\int f \leq \liminf_{n \to \infty} \int f_n.$$

[This result can be deduced from a careful examination of the proof of the Dominated Convergence Theorem. It is worth noting that the sequence $\{f_n\}$ may not converge and the stated inequality may be strict.]

Applications

1. For each $\alpha > 0$,

$$\Gamma(\alpha) := \int_0^\infty e^{-x^\alpha} \, dx = \lim_{n \to \infty} \frac{n! n^\alpha}{n! \alpha(\alpha + 1) \cdots (\alpha + n)}.$$ 

2. The Fundamental Theorem of Calculus

If the function $F$ has a bounded derivative $f$ on the closed interval $[a, b]$, then $f$ is integrable on $[a, b]$ and

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

3. Let $\{f_n\}$ be a sequence of integrable functions which converges almost everywhere to a function $f$. Suppose that there is an integrable function $g$ such that $|f| \leq g$. Then $f$ is integrable.

[Note, under the stated hypotheses, no assertion can be made about the validity of the equation

$$\int f = \lim_{n \to \infty} \int f_n.$$ ]

Nevertheless this result is of considerable importance, as will become clear in Unit 10.

4. The Riemann–Lebesgue Lemma

If $f \in L^1(\mathbb{R})$, then, for each $k \in \mathbb{R}$, the functions

$$x \mapsto f(x) \cos kx \quad \text{and} \quad x \mapsto f(x) \sin kx$$

also belong to $L^1(\mathbb{R})$ and

$$\lim_{k \to \infty} \int f(x) \cos kx \, dx = 0 = \lim_{k \to \infty} \int f(x) \sin kx \, dx.$$
Unit 10 Lebesgue Measure

The function \( \text{mid}\{, , \} \)

If \( a, b, c \in \mathbb{R} \), then \( \text{mid}\{a, b, c\} \) denotes the unique number among \( a, b, c \) that is between the other two:

\[
\text{mid}(a, b, c) = \max\{\min\{a, b\}, \min\{b, c\}, \min\{c, a\}\}
\]

\[
= \min\{\max\{a, b\}, \max\{b, c\}, \max\{c, a\}\}.
\]

Measurable functions

1. A function \( f : \mathbb{R}^k \rightarrow \mathbb{R} \) is said to be (Lebesgue) measurable if the function \( \text{mid}\{-g, f, g\} \), defined in the obvious way, is integrable for each positive function \( g \) in \( L^1(\mathbb{R}^k) \).

2. A function \( f : \mathbb{R}^k \rightarrow \mathbb{R} \) is measurable if and only if
   (a) \( \text{mid}\{-\phi, f, \phi\} \) is integrable for any positive step function \( \phi \) on \( \mathbb{R}^k \), or
   (b) \( \text{mid}\{-K\chi_I, f, K\chi_I\} \) is integrable for any positive real number \( K \) and any bounded interval \( I \).

3. (a) All continuous functions and all integrable functions are measurable.
   (b) If \( f \) and \( g \) are measurable, then so are \( |f|, \max\{f, g\}, \min\{f, g\} \) and \( af + bg (a, b \in \mathbb{R}) \).
   (c) If \( f_n \rightarrow f \) almost everywhere and \( f_n \) is measurable for \( n = 1, 2, \ldots \), then \( f \) is measurable.

Integrability

1. If \( f \) is measurable and \( |f| \leq g \), where \( g \) is integrable, then \( f \) is integrable.

2. If \( f \) is measurable and \( |f| \) is integrable, then \( f \) is integrable.

Tonelli's Theorem

If \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is measurable and one of the repeated integrals

\[
\int \left( \int |f(x, y)| \, dx \right) \, dy, \quad \int \left( \int |f(x, y)| \, dy \right) \, dx
\]

exists, then \( f \) is integrable and hence the repeated integrals

\[
\int \left( \int f(x, y) \, dx \right) \, dy, \quad \int \left( \int f(x, y) \, dy \right) \, dx
\]

both exist and are equal.

Measurable sets and measure

1. Let \( S \) be subset of \( \mathbb{R}^k \) and \( \chi_S \) be its characteristic function. Then \( S \) is said to be (Lebesgue) measurable if \( \chi_S \) is a measurable function. If \( \chi_S \) is integrable, write

\[
m(S) = \int \chi_S
\]

and if \( \chi_S \) is measurable, but not integrable, write

\[
m(S) = \infty;
\]

\( m(S) \) is called the (Lebesgue) measure of \( S \).

2. A subset \( S \) of \( \mathbb{R}^k \) is measurable if and only if \( S \cap I \) has finite measure for every bounded interval \( I \) in \( \mathbb{R}^k \).

[This result shows that the notion of measurable set could have been introduced before that of measurable function.]
3. (a) Let $S$, $T$ be measurable sets. Then so are $S \cup T$, $S \cap T$, $S \setminus T$, $S \Delta T$, and

(i) $m(S) = 0$ if and only if $S$ is a null set;
(ii) $m(S) \geq m(T)$ if $S \supset T$;
(iii) $m(S \cup T) + m(S \cap T) = m(S) + m(T)$.

(b) If $\{S_n\}$ is an increasing sequence of measurable sets and $S = \bigcup_{n=1}^{\infty} S_n$, then $S$ is measurable and

$$m(S) = \lim_{n \to \infty} m(S_n).$$

(c) If $\{S_n\}$ is a sequence of measurable sets and $S = \bigcup_{n=1}^{\infty} S_n$, then $S$ is measurable and

$$m(S) \leq \sum_{n=1}^{\infty} m(S_n);$$

moreover, if the sets $S_n$ are disjoint, then

$$m(S) = \sum_{n=1}^{\infty} m(S_n).$$

4. A function $f : \mathbb{R}^k \to \mathbb{R}$ is measurable if and only if for each real number $c$, the set

$$A_c = \{x \in \mathbb{R}^k : f(x) \geq c\}$$

is measurable.

[It was noted above that the notion of measurable set could have been introduced before that of measurable function. This result shows that measurable functions could then have been defined in terms of measurable sets. This was the order in which Lebesgue developed his theory.]

Simple functions

1. A function $\phi$ of the form

$$\phi = c_1 x_{S_1} + c_2 x_{S_2} + \cdots + c_r x_{S_r},$$

where $r \in \mathbb{Z}^+$, $S_1, S_2, \ldots, S_r$ are measurable sets and $c_1, c_2, \ldots, c_r$ are real numbers, is called a simple function (or generalized step function).

[Note that a simple function need not be integrable.]

2. A measurable function $f : \mathbb{R}^k \to \mathbb{R}$ may be expressed as the limit, everywhere, of a sequence of simple functions; if $f$ is positive, we may choose an increasing sequence of simple functions converging everywhere to $f$.

Measureable functions

1. If $h : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f, g : \mathbb{R}^k \to \mathbb{R}$ are measurable, then the composite function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x) = h(f(x), g(x)) \quad (x \in \mathbb{R}^k)$$

is also measurable.

2. If $f, g : \mathbb{R}^k \to \mathbb{R}$ are measurable, then their product $fg$ is also measurable.
Measurable and non-measurable sets: examples

1. All open sets in \( \mathbb{R}^k \) and all closed sets in \( \mathbb{R}^k \) are measurable. \( \text{Proposition 6.3.1, W page 135} \)
2. There is a non-measurable subset of \( \mathbb{R} \). \( \text{W page 131} \)
   [The construction of such a set depends on the Axiom of Choice.]

Unit 11 Convergence and Normed Spaces

Norms, semi-norms, normed spaces

1. A norm on a real vector space \( V \) is a function \( \| \cdot \| \) of \( V \) into \( \mathbb{R} \) having the following properties:
   \[
   \begin{align*}
   (N1) & \quad \text{for each } v \in V, \|v\| \geq 0; \\
   (N2) & \quad \|v\| = 0 \text{ if and only if } v = 0, \text{ the zero vector}; \\
   (N3) & \quad \text{for each } v \in V \text{ and } \alpha \in \mathbb{R}, \|\alpha v\| = |\alpha| \|v\|; \\
   (N4) & \quad \text{for all } v_1, v_2 \in V, \|v_1 + v_2\| \leq \|v_1\| + \|v_2\|.
   \end{align*}
   \]
   A function having properties (N1), (N3) and (N4) is called a semi-norm on \( V \).
   A vector space with a norm defined on it is called a normed space.
   [According to the above definition, a normed space is a composite object, consisting of a vector space and a norm on that vector space. Nevertheless, when it is unlikely to cause confusion, we abuse notation and denote a normed space by the same symbol as the underlying vector space.]

2. The modulus function is a norm on \( \mathbb{R} \).

The space \( L^p \)

1. Let \( p \) be a real number satisfying \( p \geq 1 \). Denote by \( L^p \) the set of all measurable functions \( f \) such that \( |f|^p \) is integrable, and, for each \( f \in L^p \), define
   \[
   \|f\|_p = \left\{ \int |f|^p \right\}^{1/p}.
   \]
   [If \( f \) is measurable and \( |f| \) is integrable, then \( f \) is integrable. Thus the above definition does not conflict with the earlier definition of \( L^1 \).]

2. For each \( p \geq 1 \), \( L^p \) is a real vector space and \( \| \cdot \|_p \) is a semi-norm on \( L^p \). Also, if \( f \in L^p \), then
   \[
   \|f\|_p = 0 \quad \text{if and only if } f = 0 \text{ almost everywhere}.
   \]
   [If functions \( f, g \) such that \( f = g \) almost everywhere are not distinguished, then \( \| \cdot \|_p \) is a norm on \( L^p \). There is a more formal discussion of this difficulty in the appendix to this unit.]

Convergence

1. A sequence \( \{v_n\} \) of elements of a normed space \( V \) is said to converge in \( V \) to an element \( v \) of \( V \) if
   \[
   \lim_{n \to \infty} \|v_n - v\| = 0,
   \]
   \( \| \cdot \| \) being the norm on \( V \). \( \text{W page 164} \)
2. According to the above definition, a sequence \( \{f_n\} \) of functions in \( L^p \) converges in \( L^p \) to a function \( f \) in \( L^p \) if
\[
\lim_{n \to \infty} ||f_n - f||_p = 0.
\]
This kind of convergence is often called strong convergence or convergence in mean (of order \( p \)).

3. Let \( \{f_n\} \) be a sequence of functions in \( L^1 \), converging a.e. to a function \( f \), and obeying either the conditions of the Monotone Convergence Theorem or the conditions of the Dominated Convergence Theorem. Then \( \{f_n\} \) converges to \( f \) in \( L^1 \).

[It is worth noting that, in general, convergence in \( L^1 \) neither implies nor is implied by pointwise convergence a.e.]

### Cauchy sequence

A sequence \( \{v_n\} \) of elements of a normed space \( V \) is called a **Cauchy sequence** if
\[
\lim_{m,n \to \infty} ||v_n - v_m|| = 0,
\]
that is, given any \( \varepsilon > 0 \), there is an \( N \) in \( \mathbb{Z}^+ \) such that,

\[
\text{for all } m, n \geq N, \quad ||v_n - v_m|| < \varepsilon.
\]

### Completeness

1. It is easily shown that every convergent sequence in a normed space \( V \) is Cauchy. If the converse is true, that is if every Cauchy sequence of elements of \( V \) converges to an element of \( V \), then \( V \) is said to be complete.

2. The normed space \( \mathbb{R} \) is complete in this sense.

3. Let \( a_n \in L^p \) for \( n = 1, 2, \ldots \). If \( \sum ||a_n||_p \) converges in \( \mathbb{R} \), then \( \sum a_n \) converges in \( L^p \) to a function \( s \). Moreover, \( \sum a_n(x) \) converges to \( s(x) \) for all \( x \).

4. For each \( p \geq 1 \), the normed space \( L^p \) is complete.

5. Let \( \{f_n\} \) be a Cauchy sequence in \( L^p \). Then there is a subsequence of \( \{f_n\} \) which converges pointwise almost everywhere to a function \( f \), where \( f \in L^p \) and \( \{f_n\} \) converges strongly to \( f \).

[This result can be deduced from the proof of the above result.]

---

# Unit 12 Hilbert Spaces

### Inner products, Euclidean spaces

1. An **inner product** on a real vector space \( V \) is a function of \( V \times V \) into \( \mathbb{R} \) which associates with the pair of vectors \( (v, w) \) the real number \( v \cdot w \) in such a way that:
   
   - (a) \((v_1 + v_2) \cdot w = v_1 \cdot w + v_2 \cdot w \quad (v_1, v_2, w \in V)\);
   - (b) \(v \cdot w = w \cdot v \quad (v, w \in V)\);
   - (c) \((\lambda v) \cdot w = \lambda(v \cdot w) \quad (v, w \in V; \lambda \in \mathbb{R})\);
   - (d) \(v \cdot v \geq 0 \) and \( v \cdot v = 0 \) if and only if \( v = 0 \) \((v \in V)\).
A real vector space with an inner product defined on it is called a Euclidean space.

[As with normed spaces, we shall often denote a Euclidean space by the same symbol as the underlying vector space.]

2. Let $V$ be a Euclidean space. Then

(a) Schwarz's (or Cauchy-Schwarz's) Inequality is

$$|v \cdot w| \leq (v \cdot v)^{1/2} (w \cdot w)^{1/2} \quad (v, w \in V)$$

and

(b) $|| \cdot ||$ defined by

$$||v|| = (v \cdot v)^{1/2} \quad (v \in V)$$

is a norm on $V$.

[Part (b) shows that every Euclidean space can be considered as a normed space.]

3. If $V$ is a vector space with a norm derived from an inner product, then the Parallelogram Law holds, that is, for all $v, w \in V$,

$$||v + w||^2 + ||v - w||^2 = 2(||v||^2 + ||w||^2).$$

[The Parallelogram Law is, in fact, a sufficient condition for a norm on a vector space to be derived from an inner product.]

Orthogonal and orthonormal sets

Let $V$ be a Euclidean space. Then

(a) two vectors $v, w$ in $V$ are said to be orthogonal if $v \cdot w = 0$;

(b) a subset $A$ of $V$ is said to be orthogonal if any two distinct vectors in $A$ are orthogonal;

(c) a subset $A$ of $V$ is said to be orthonormal if it is orthogonal and, for each $v \in A$, $||v|| = 1$.

Linearly independent, span, Hamel bases

1. Let $V$ be a vector space and $A$ be a subset of $V$. $A$ is said to be linearly independent if, for every finite subset $\{v_1, v_2, \ldots, v_n\}$ of $A$,

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, implies

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

If $A$ is not linearly independent, it is said to be linearly dependent.

The span of $A$ is the set of all finite linear combinations of vectors from $A$. The set $A$ is said to be spanning for $V$ if the span of $A$ coincides with $V$.

A Hamel basis for $V$ is a set of vectors in $V$ which is both linearly independent and spanning for $V$.

[The span of $A$ is the smallest subspace of $V$, which contains the set $A$. A finite Hamel basis is usually referred to simply as a basis.]

2. An orthogonal set of non-zero vectors in a Euclidean space $V$ is linearly independent.
3. Let \( V \) be a finite-dimensional Euclidean space with an orthonormal (Hamel) basis \( \{e_1, e_2, \ldots, e_k\} \).

(a) Each vector \( v \) in \( V \) can be uniquely expressed in the form
\[
v = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_k e_k;
\]
the coefficients are given by
\[
\lambda_i = v \cdot e_i, \quad i = 1, \ldots, k.
\]

(b) For all \( v, w \in V \),
\[
v \cdot w = \sum_{i=1}^{k} (v \cdot e_i)(w \cdot e_i);
\]
in particular,
\[
||v||^2 = \sum_{i=1}^{k} (v \cdot e_i)^2.
\]

(c) If \( \mathbb{R}^k \) is considered as a Euclidean space with respect to the standard inner product
\[
(a_1, a_2, \ldots, a_k) \cdot (b_1, b_2, \ldots, b_k) = \sum_{i=1}^{k} a_i b_i,
\]
then the function \( \theta : V \to \mathbb{R}^k \) given by
\[
\theta(v) = (v \cdot e_1, v \cdot e_2, \ldots, v \cdot e_k) \quad (v \in V)
\]
is a one-one linear transformation of \( V \) onto \( \mathbb{R}^k \) which preserves inner products, that is, for all \( v, w \in V \),
\[
\theta(v) \cdot \theta(w) = v \cdot w.
\]
[Such a function \( \theta \) is called a (Euclidean space) isomorphism of \( V \) onto \( \mathbb{R}^k \).

4. Let \( \{e_1, e_2, \ldots, e_n\} \) be an orthonormal set of vectors in a Euclidean space \( V \) and \( v \in V \). Then, for any real numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \),
\[
\left| |v - \sum_{i=1}^{n} \lambda_i e_i| \right| \geq \left| |v - \sum_{i=1}^{n} (v \cdot e_i)e_i| \right|
\]

The space \( \ell^2 \)

1. Denote by \( \ell^2 \) the set of all real sequences \((a_1, a_2, \ldots)\) for which the series \( \sum_{i=1}^{\infty} a_i^2 \) is convergent.

(a) If \((a_1, a_2, \ldots), (b_1, b_2, \ldots) \in \ell^2 \) and \( \lambda \in \mathbb{R} \), then each of the series
\[
\sum_{i=1}^{\infty} (a_i + b_i)^2, \quad \sum_{i=1}^{\infty} (\lambda a_i)^2, \quad \sum_{i=1}^{\infty} a_i b_i
\]
is convergent.

(b) If \( a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in \ell^2 \) and \( \lambda \in \mathbb{R} \), define
\[
a + b = (a_1 + b_1, a_2 + b_2, \ldots) \quad \text{and} \quad \lambda a = (\lambda a_1, \lambda a_2, \ldots).
\]
Under these algebraic operations, \( \ell^2 \) becomes a real vector space.

(c) If \( a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in \ell^2 \), define
\[
a \cdot b = \sum_{i=1}^{\infty} a_i b_i.
\]
This equation defines an inner product on \( \ell^2 \).
[Part (a) implies that \( \ell^2 \) is closed with respect to the algebraic operations defined in part (b). Part (a) also implies that, if \( a, b \in \ell^2 \), the series defining \( a \cdot b \) in part (c) is convergent. The Euclidean space \( \ell^2 \) is an infinite-dimensional analogue of the space \( \mathbb{R}^k \).]

2. Write \( \hat{e}_1 = (1, 0, 0, \ldots) \), \( \hat{e}_2 = (0, 1, 0, \ldots) \) and, in general, \( \hat{e}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}, \ldots) \) for each \( i \in \mathbb{Z}^+ \), where

\[
\delta_{ij} = 1 \quad \text{and} \quad \delta_{ij} = 0 \quad \text{for each} \quad j \in \mathbb{Z}^+ \setminus \{i\}.
\]

Then \( \{\hat{e}_1, \hat{e}_2, \ldots\} \) is an orthonormal subset of \( \ell^2 \) and, for each \( a \in \ell^2 \), the series

\[
\sum_{i=1}^{\infty} (a \cdot \hat{e}_i) \hat{e}_i
\]

converges to \( a \) in the norm derived from the inner product on \( \ell^2 \).

### Schauder bases

1. Let \( N \) be a normed space and \( S = \{s_1, s_2, \ldots\} \) be a countable infinite subset of \( N \). Then \( S \) is called a Schauder basis for \( N \) if, for each \( x \in N \), there are unique real numbers \( a_1, a_2, \ldots \) such that

\[
\sum_{i=1}^{\infty} a_is_i
\]

converges in \( N \) to \( x \).

2. The set \( \{\hat{e}_1, \hat{e}_2, \ldots\} \), defined above, is a Schauder basis for \( \ell^2 \).

### Dense and total sets: separable spaces

1. Let \( N \) be a normed space. (a) A subset \( A \) of \( N \) is said to be dense in \( N \) if, for each \( x \in N \) and \( \varepsilon > 0 \), there is an element \( a \) in \( A \) such that \( ||x - a|| < \varepsilon \).

(b) A subset \( C \) of \( N \) is said to be total in \( N \) if \( (C) \), the span of \( C \), is dense in \( N \).

(c) \( N \) is said to be separable if it has a countable dense subset.

2. A Schauder basis for a normed space \( N \) is total in \( N \).

3. If a normed space \( N \) has a countably infinite total subset (for example a Schauder basis), then \( N \) is separable.

4. Let \( S = \{e_1, e_2, \ldots\} \) be a countably infinite, total, orthonormal subset of a Euclidean space \( E \). Suppose that, for each \( x \in E \), the series

\[
\sum_{i=1}^{\infty} (x \cdot e_i)e_i
\]

converges in \( E \), to \( s(x) \) say. Then \( S \) is a Schauder basis for \( E \) and, for each \( x \in E \), \( s(x) = x \), that is

\[
x = \sum_{i=1}^{\infty} (x \cdot e_i)e_i.
\]

### Fourier coefficients, Fourier series

1. Let \( E \) be a Euclidean space with an orthonormal Schauder basis \( \{e_1, e_2, \ldots\} \) and \( x \in E \). Then the numbers \( x \cdot e_i \) are called the Fourier coefficients of \( x \) and

\[
\sum_{i=1}^{\infty} (x \cdot e_i)e_i
\]

is called the Fourier series of \( x \) with respect to this basis.
2. Let \( S = \{ e_1, e_2, \ldots \} \) be a countably infinite orthonormal subset of a Euclidean space \( E \).

(a) **Bessel's Inequality**  For each \( x \in E \), the series \( \sum_{i=1}^{\infty} (x \cdot e_i)^2 \) is convergent and

\[
\sum_{i=1}^{\infty} (x \cdot e_i)^2 \leq ||x||^2.
\]

(b) **Parseval's Equation**  The set \( S \) is a Schauder basis for \( E \) if and only if, for each \( x \in E \),

\[
\sum_{i=1}^{\infty} (x \cdot e_i)^2 = ||x||^2.
\]

(c) If \( S \) is a Schauder basis for \( E \), then, for all \( x, y \in E \),

\[
x \cdot y = \sum_{i=1}^{\infty} (x \cdot e_i)(y \cdot e_i).
\]

**Hilbert space**

1. A complete Euclidean space is called a **Hilbert space**.

2. \( \mathbb{R}^k \) is a Hilbert space.

3. Let \( H \) be an infinite-dimensional Hilbert space and \( S = \{ e_1, e_2, \ldots \} \) be a countably infinite, orthonormal subset.

(a) If \( a = (a_1, a_2, \ldots) \) is a sequence of real numbers, then \( \sum_{i=1}^{\infty} a_i e_i \) converges in \( H \) if and only if \( a \in \ell^2 \).

(b) If \( S \) is total, then \( S \) is a Schauder basis for \( H \).

**Separable Hilbert spaces**

1. An infinite-dimensional Hilbert space has a Schauder basis if and only if it is separable. Moreover, a separable infinite-dimensional Hilbert space has an orthonormal Schauder basis.

2. An orthonormal subset of a separable Hilbert space is either finite or countably infinite.

3. Let \( H \) be a separable infinite-dimensional Hilbert space. Then \( H \) is **isomorphic to** \( \ell^2 \), that is there is a one-one linear transformation \( \varphi \) of \( H \) onto \( \ell^2 \) such that, for all \( x, y \in H \),

\[
\varphi(x) \cdot \varphi(y) = x \cdot y.
\]

[This result is the infinite-dimensional analogue of an earlier result. It implies that, up to isomorphism, there is at most one separable infinite-dimensional Hilbert space. But is there any such space? The next result answers this question.]

**The Hilbert space \( L^2(\mathbb{R}) \)**

1. If \( f, g \in L^2(\mathbb{R}) \), then \( fg \in L^1(\mathbb{R}) \) and the equation

\[
f \cdot g = \int fg
\]

defines an inner product on \( L^2(\mathbb{R}) \). The derived norm is just the norm \( || \cdot ||_2 \), discussed in Unit 11. As we saw in that unit, \( L^2(\mathbb{R}) \) is complete with respect to that norm, that is \( L^2(\mathbb{R}) \) is a Hilbert space. \( L^2(\mathbb{R}) \) is infinite-dimensional and separable.
2. Let $E$ be a Euclidean space, $H$ be a Hilbert space and $\sigma$ be a one-one linear transformation of $H$ onto $E$ which preserves inner products. Then $E$ is also a Hilbert space. In addition, if $H$ is separable, then so is $E$.

[According to earlier results, there is a one-one linear transformation of $L^2(\mathbb{R})$ onto $\ell^2$ which preserves inner products. The above result therefore implies that $\ell^2$ is a separable Hilbert space. In fact, we have already pointed out that $\ell^2$ has a Schauder basis, and so is separable. It is not difficult to prove directly that $\ell^2$ is a Hilbert space.]
The functions $D_n$, defined in part 1, are called the Dirichlet kernels. The main tool in the proof of part 2 (b) is the Riemann–Lebesgue Lemma—see Unit 9.

$f(x + 0)$, $f(x - 0)$

1. For any $f : \mathbb{R} \to \mathbb{R}$, we denote the one-sided limits

$$\lim_{t \to 0^+} f(x + t) \quad \text{and} \quad \lim_{t \to 0^-} f(x - t)$$

(if they exist) by $f(x + 0)$ and $f(x - 0)$ respectively.

[Both of these limits exist if $f$ is a step function, $f$ is monotonic or $f$ is the difference of two monotonic functions.]

2. Let $\phi$ be a step function and $f$ be the function of period $2\pi$ which coincides with $\phi$ on $(-\pi, \pi]$. Then the Fourier series of $f$ at any point $x$ converges to the sum

$$\frac{1}{2}(f(x + 0) + f(x - 0)).$$

The space $L^2(-\pi, \pi)$

(a) Denote by $L^2(-\pi, \pi)$ the set of measurable functions $f$ on $\mathbb{R}$ for which $fX(-\pi, \pi) \in L^2(\mathbb{R})$. All the arguments applied to $L^2(\mathbb{R})$ in Units 11 and 12 also apply to $L^2(-\pi, \pi)$. Thus $L^2(-\pi, \pi)$ is a Hilbert space.

(b) Define the functions $e_1, e_2, \ldots$ by

$$e_1(x) = \frac{1}{\sqrt{2\pi}},$$

$$e_{2k}(x) = \frac{1}{\sqrt{\pi}} \sin kx \quad (k = 1, 2, \ldots),$$

$$e_{2k+1}(x) = \frac{1}{\sqrt{\pi}} \cos kx \quad (k = 1, 2, \ldots).$$

$\{e_1, e_2, \ldots\}$ is an orthonormal subset of $L^2(-\pi, \pi)$.

(c) If $f \in L^2(-\pi, \pi)$, then $f \in L^1(-\pi, \pi)$ and the Fourier series of $f$, as defined above, is

$$\sum_{i=1}^{\infty} (f \cdot e_i)e_i(x).$$

(d) $\{e_1, e_2, \ldots\}$ is a total subset of $L^2(-\pi, \pi)$.

(The key steps in the proof of part (d) are as follows.

(i) If $\phi$ is a step function, which vanishes outside $(-\pi, \pi)$, then Bessel’s Inequality, the completeness of $L^2(-\pi, \pi)$ and Lemma T.12.2.3(2) imply that $\sum_{i=1}^{\infty} (\phi \cdot e_i)e_i$ converges in $L^2(-\pi, \pi)$, to $g$ say: in fact, the above result implies that $g = \phi$ a.e.

(ii) The set of step functions, which vanish outside $(-\pi, \pi)$, is dense in $L^2(-\pi, \pi)$. This is easily deduced from Stage 1 of the proof of Theorem T.12.3.1. The word ‘total’ in part (d) is sometimes replaced by ‘complete’.)

Strong convergence

The above result shows that $\{e_1, e_2, \ldots\}$ is an orthonormal Schauder basis for $L^2(-\pi, \pi)$, and so, by results of Unit 12, the Fourier series of any function $f$ in $L^2(-\pi, \pi)$ converges strongly to $f$. 

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Theorem 7.6.2, W page 207
Bounded variation

A real-valued function $f$ is said to have *bounded variation* on an interval $[a, b]$ if, on that interval, $f$ is the difference of two increasing functions.

[Strictly speaking this is a characterization of functions having bounded variations on an interval, and not the definition of such functions.]

Pointwise convergence

*Jordan's Theorem*  

Let $f \in L^1(-\pi, \pi)$ and have period $2\pi$. If $f$ has bounded variation on an interval $[x-r, x+r]$, where $0 < r \leq \pi$, then the Fourier series of $f$ converges at $x$ to the sum  

$$\frac{1}{2}(f(x + 0) + f(x - 0)).$$

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