THE LEBESGUE INTEGRAL

Unit 13
Fourier Series
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Unit 13
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Prepared by the Course Team
Set Book


It is essential to have this book; the course is based on it and will not make sense without it.

This unit is based on Section 7.6 of the set book, pages 202 to 211.

Bibliography

The following book is referred to quite frequently, and is useful though not essential.


Conventions

Before starting work on this text, please read the *Guide to the Course*.

The set book is referred to as *Weir*, and the above book *Calculus*, by M. Spivak, is referred to as *Spivak*. 
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**Introduction**

The previous two units, *Unit 11, Convergence and Normed Spaces* and *Unit 12, Hilbert Space*, have prepared the ground for the central problem of this unit, namely the problem of the convergence of Fourier series. Let us begin with a brief overview of the classical Fourier series construction.

By the classical Fourier series for a function \( f : \mathbb{R} \to \mathbb{R} \) we mean the infinite series

\[
\frac{1}{2}a_0 + \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right)
\]

of trigonometric functions, where the coefficients \( a_k \) and \( b_k \) are given by

\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt \quad (k = 0, 1, 2, \ldots),
\]

\[
b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt \quad (k = 1, 2, \ldots).
\]

The question we must answer is: does the Fourier series converge, and, if so, does it converge to the original function \( f \)? As we saw in *Unit 11*, there is more than one sense in which we may consider a sequence of functions to 'converge', so we must phrase this question more precisely. In so doing, we see that there are in fact two different, though related, convergence problems to be tackled.

A. Does the Fourier series for \( f \) converge to \( f \) in the sense of a suitable norm?

B. Does the Fourier series for \( f \) converge to \( f \) pointwise?

In *Unit 12, Hilbert Space* we saw that the appropriate context in which to discuss the convergence of Fourier series, in general, is a *Hilbert space*. From that unit we know, that if \( \{e_1, e_2, \ldots\} \) is an orthonormal Schauder basis for the separable Hilbert space \( H \), then the abstract Fourier series of an element \( f \) of \( H \), with respect to this basis, \( \sum_{i=1}^{\infty} (f \cdot e_i) e_i \), converges to \( f \) in the sense of the norm of \( H \). Therefore in order to answer Question A we need only describe the appropriate Hilbert space —which turns out to be \( L^2(-\pi, \pi) \)— and show that the trigonometric functions

\[
x \mapsto \cos kx \quad (k \geq 0), \quad \text{and} \quad x \mapsto \sin kx \quad (k \geq 1),
\]

form a complete orthogonal set in this space. By Theorem T.12.2.1 of *Unit 12* we may then normalize these functions to obtain an orthonormal Schauder basis \( \{e_1, e_2, e_3, \ldots\} \) with

\[
e_1 : x \mapsto 1/\sqrt{2\pi},
\]

\[
e_{2k} : x \mapsto (1/\sqrt{\pi}) \sin kx, \quad (k = 1, 2, \ldots),
\]

\[
e_{2k+1} : x \mapsto (1/\sqrt{\pi}) \cos kx, \quad (k = 1, 2, \ldots).
\]

Substituting these functions into the Fourier series

\[
(f \cdot e_1)e_1 + (f \cdot e_2)e_2 + (f \cdot e_3)e_3 + \cdots,
\]

we obtain the series

\[
\left( (1/\sqrt{2\pi}) \int_{-\pi}^{\pi} f(t) \, dt \right) \left( (1/\sqrt{2\pi}) \right) + \left( (1/\sqrt{\pi}) \int_{-\pi}^{\pi} f(t) \sin t \, dt \right) \left( (1/\sqrt{\pi}) \sin x \right) + \left( (1/\sqrt{\pi}) \int_{-\pi}^{\pi} f(t) \cos t \, dt \right) \left( (1/\sqrt{\pi}) \cos x \right) + \cdots,
\]

which is identical to the classical Fourier series described at the beginning of this Introduction. From our general theory, this is guaranteed to converge to \( f \) in the sense of the norm on \( L^2(-\pi, \pi) \), that is, *strongly*. It is worth emphasizing here that we have assumed \( f \in L^2(-\pi, \pi) \). We have not shown that the series converges to an element of \( L^2(-\pi, \pi) \) if we take \( f \) merely to be a function for which the integrals \( f \cdot e_k \) are finite.

The simplicity of this answer contrasts with the more difficult problem of pointwise convergence posed in Question B. This difficulty is partly due to the fact that the elaborate machinery constructed in the previous units to deal with norm
convergence is largely irrelevant here. We are therefore compelled to introduce some
new machinery specially geared to the discussion of pointwise convergence, namely
the Dirichlet kernels, $D_n(x)$.

Note that we are using the word kernel as synonymous with function, usually
appearing as a multiplicative factor in an integrand. The Dirichlet functions enable
us to express the $n$th partial sum $s_n$ of the Fourier series for $f$ in the simple form

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t)D_n(t)dt \quad (n = 1, 2, \ldots).$$

With the aid of the Dirichlet kernels, we are able to prove Jordan's Theorem which
establishes the pointwise convergence of the Fourier series of a general class
Of functions called functions of bounded variation. (For the purposes of this unit, a
function $f$ has bounded variation on $[a, b]$ if $f$ can be expressed as the difference
of two functions which are increasing on that interval.) This class includes piecewise
smooth functions, and so Jordan's Theorem provides us with a proof of the pointwise
convergence of the Fourier series of a function piecewise smooth on $(-\pi, \pi)$. This
is a non-trivial result.

In Section 1 we describe the Hilbert space $L^2(-\pi, \pi)$ and introduce the Dirichlet
kernels $D_n(x)$; we demonstrate the connection between these kernels and the
pointwise convergence of a Fourier series in $L^2(-\pi, \pi)$.

In Section 2 we prove the result we need to answer Question A, namely that
the trigonometric functions form a complete set of functions in $L^2(-\pi, \pi)$. The proof
of this result makes use of the Dirichlet kernels and pointwise convergence, and
provides a link between the problems of norm convergence and pointwise convergence.

In Section 3 we prove Jordan's Theorem concerning the pointwise convergence of the
Fourier series of functions of bounded variation. This gives a sufficient condition on
the function $f$ in order to answer our Question B above in the affirmative.

All the Hilbert spaces we consider in this unit are separable and therefore have an
orthonormal Schauder basis which is assumed without comment throughout.

1 Dirichlet kernels

Our first task is to find the right space in which to discuss classical Fourier series.
The key observation is to note that in constructing the Fourier series of a function $f$
we integrated over the interval $[-\pi, \pi]$; we ignored the behaviour of $f$ outside this
interval. This means that we are really interested in functions of the form $f\chi_{[-\pi, \pi]}$,
more specifically, functions in $L^1[-\pi, \pi]$.

We also wanted our space of functions to be a Hilbert space, which suggests that we
look at the space

$$L^2[-\pi, \pi] = \{f \text{ measurable} : f\chi_{[-\pi, \pi]} \in L^2\},$$

which, because

$$\chi_{[-\pi, \pi]}^2 = \chi_{[-\pi, \pi]},$$

can also be written as

$$L^2[-\pi, \pi] = \{f \text{ measurable} : f^2 \in L^1[-\pi, \pi]\}.$$

It is straightforward to show that all the arguments which we applied to $L^2$ in
Units 11 and 12 also apply to $L^2[-\pi, \pi]$. Therefore, $L^2[-\pi, \pi]$ is a Hilbert space
with all that that implies. In fact, if $I$ is any interval in $\mathbb{R}$ with non-zero measure,
then $L^2(I)$, defined in an analogous way to $L^2[-\pi, \pi]$, is also a Hilbert space.
The end-points of the interval \([-\pi, \pi]\), having measure zero, do not contribute to the integrals in the computation of the Fourier coefficients, so we could equally well consider the spaces

\[ L^2(-\pi, \pi), \quad L^2[-\pi, \pi], \quad L^2(-\pi, \pi). \]

However, Weir points out in the first part of the reading passage that these spaces are all equal and, though \(L^2(-\pi, \pi)\) is the right space for our discussion, we shall use \(L^2(-\pi, \pi)\) as the notation for this space.

The main part of the following reading passage introduces the Dirichlet kernels as functions from \(\mathbb{R}\) to \(\mathbb{R}\) given by

\[ D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx \quad (n = 1, 2, \ldots). \]

These functions arise naturally when we look at the partial sums of the Fourier series for a function \(f\). Our main task in this section is to explore the connection between these kernels and the pointwise converge of the Fourier series.

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**Read** Weir: Chapter 7, Subsection 7.6, page 202 to page 206, line 3.

**Notes**

1. **Page 202, line -7 to page 203, line 4.** The point that is being made here is that, given any function \(f\) for which the appropriate integrals exist, we can always write down the Fourier series of \(f\),

\[ \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \]

Now this series is periodic, so, for any chance for it to converge pointwise to \(f\) everywhere, \(f\) itself must be periodic. Therefore, the type of function we have to consider can be defined arbitrarily on \((-\pi, \pi]\) but then is extended by periodicity to the whole of \(\mathbb{R}\). For example, if we use the identity function \(g\), given by \(g(x) = x\), then the Fourier series for \(g\),

\[ 2 \sum_{k=1}^{\infty} \frac{1}{k}(-1)^{k+1} \sin kx, \]

converges pointwise almost everywhere to the function illustrated in Figure 68 (Weir: page 203) which agrees with \(g\) only on \((-\pi, \pi]\). However, it is the behaviour of a function in \((-\pi, \pi]\) only which determines its Fourier series, so the Fourier series for the function defined by Figure 68 is precisely the same as that for the function \(g\). Therefore, we might just as well have started from the function in Figure 68 in the first place.

2. **Page 204, lines 1 to 4.** Weir is using the result in Exercise 5 of Unit 10, Lebesgue Measure (Weir: page 124, Exercise 6), namely that if \(f^2\) and \(g^2\) are integrable functions then so is the function \(fg\). In the present context this means that

\[ \text{if } f, g \in L^2(-\pi, \pi), \text{ then } fg \in L^1(-\pi, \pi). \]

3. **Page 204, line -8.** In the abstract setting of Fourier series in a Hilbert space, the series

\[ \sum_{i=1}^{\infty} (x \cdot e_i)^2 \leq ||x||^2 \]

converges, by Bessel's inequality, where the \((x \cdot e_i)\) are the Fourier coefficients relative to the orthonormal set of vectors \(\{e_1, e_2, \ldots\}\). This implies that the sequence \(\{(x \cdot e_i)^2\}\), and hence the sequence \(\{(x \cdot e_i)\}\), converges to zero. Here we have a special case of this result.
Recall that, for any \(a, b \in \mathbb{R}\),
\[
\cos(a - b) = \cos a \cos b + \sin a \sin b.
\]
Here this result is being used with \(a = kt\) and \(b = kx\).

Weir is using the fact that if \(F\) has period \(2\pi\), then
\[
\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi+\pi}^{\pi} f(t)\,dt,
\]
which is a simple translation of the limits of integration by any real number \(x\).

We may then 'translate' the function \(F\) to get
\[
\int_{-\pi}^{\pi} F(t)\,dt = \int_{-\pi + x}^{\pi + x} F(t + x)\,dt.
\]

Translating back the limits of integration (by \(-x\)) gives
\[
\int_{-\pi}^{\pi} F(t)\,dt = \int_{-\pi}^{\pi} F(t + x)\,dt.
\]
(Alternatively, we may deduce this equality immediately by substituting \(t' = t - x\) in the first equation.)

Put
\[
F(t) = f(t)D_n(t - x)
\]
to get Weir's result.

It does not matter what the function \(s\) is; the proposition gives a necessary and sufficient condition for the sequence of partial sums \(s_n(x)\) to converge to \(s(x)\).

Subtracting the identities
\[
\sin(a + b) = \sin a \cos b + \sin b \cos a
\]
\[
\sin(a - b) = \sin a \cos b - \sin b \cos a,
\]
we obtain
\[
\sin(a + b) - \sin(a - b) = 2\sin b \cos a,
\]
for any \(a, b \in \mathbb{R}\). This identity is being used here with \(a = kx, b = x/2\).

**Exercise 1**
Show that the trigonometric functions
\[
x \mapsto \cos kx \text{ and } x \mapsto \sin kx,
\]
where \(k = 0, 1, 2, \ldots\), form an orthogonal set. Normalize the set.

**Solution**
We must show that the following integrals are all zero:
\[
\int_{-\pi}^{\pi} \cos kx \cos lx\,dx, \quad k \neq l,
\]
\[
\int_{-\pi}^{\pi} \sin kx \sin lx\,dx, \quad k \neq l,
\]
\[
\int_{-\pi}^{\pi} \cos kx \sin lx\,dx.
\]

The evaluations are all similar, and use the trigonometric identities:
\[
\cos kx \cos lx = \frac{1}{2} \left( \cos(k + l)x + \cos(k - l)x \right),
\]
\[
\sin kx \sin lx = \frac{1}{2} \left( \cos(k - l)x - \cos(k + l)x \right),
\]
\[
\cos kx \sin lx = \frac{1}{2} \left( \sin(k + l)x - \sin(k - l)x \right).
\]
We evaluate the first integral as an example:

\[ \int_{-\pi}^{\pi} \cos kx \cos lx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos(k+l)x + \cos(k-l)x) \, dx \]

\[ = \frac{1}{2} \left[ \frac{1}{k+l} \sin(k+l)x + \frac{1}{k-l} \sin(k-l)x \right]_{-\pi}^{\pi} = 0, \quad k \neq l, \]

since \( \sin n\pi = 0 \), for \( n \in \mathbb{Z} \).

Similarly for the other two integrals; the functions therefore form an orthogonal set.

In more general notation, we have shown that a set of vectors \( \{f_1, f_2, \ldots\} \) obeys

\[ f_i \cdot f_j = 0 \quad (i \neq j). \]

We normalize the set by defining the new set \( \{e_1, e_2, \ldots\} \), where

\[ e_i = f_i / \|f_i\| \quad (i = 1, 2, \ldots). \]

This is clearly now an orthonormal set. In our case

\[ \|\cos kx\|^2 = \int_{-\pi}^{\pi} (\cos kx)^2 \, dx \]

\[ = \int_{-\pi}^{\pi} \left( \frac{1}{2} + \frac{1}{2} \cos 2kx \right) dx \quad (k \neq 0) \quad (\text{using the identity given above}) \]

\[ = \pi \quad (k \neq 0) \text{ for } k = 0 \text{ we get } 2\pi. \]

Similarly,

\[ \|\sin kx\| = \sqrt{\pi}, \quad k \neq 0. \]

Therefore the set

\[ \{x \mapsto (1/\sqrt{2\pi}), x \mapsto (1/\sqrt{\pi}) \cos kx, x \mapsto (1/\sqrt{\pi}) \sin kx : k = 1, 2, \ldots\} \]

is an orthonormal set of functions in \( L^2(-\pi, \pi) \). The important fact that these trigonometric functions form a complete set for \( L^2(-\pi, \pi) \) — and thus that \( \{e_1, e_2, \ldots\} \) is an orthonormal Schauder basis for \( L^2(-\pi, \pi) \) — is the main result of Section 2.

**Exercise 2**

Let \( x \in (-\pi, \pi) \) and let \( f \) be the identity function \( f(t) = t \), for all \( t \in (-\pi, \pi) \). Use Proposition 7.6.1 (Weir: page 205) to show that the Fourier series for \( f \) converges to \( x \) at \( x \).

**Solution** With the particular choice of \( f \) as the identity function and \( s(x) = x \),

\[ g(x,t) = x + t + x - t - 2x = 0. \]

Hence

\[ \int_{0}^{\pi} g(x,t)D_n(t) \, dt = 0 \]

for all \( n \), and the required result follows from Proposition 7.6.1.

**Exercise 3**

Let \( x \in (0, \pi) \) and \( I = (-\pi, \pi) \). Show that the Fourier series for \( \chi_I \in L^1(-\pi, \pi) \) converges to \( \frac{1}{2} \) at \( x \).

**Solution** As in Exercise 2, we use Proposition 7.6.1, but this time

\[ g(x,t) = \chi_I(x+t) + \chi_I(x-t) - 1. \]

Now

\[ \chi_I(x+t) = \begin{cases} 1 & \text{if } t \in (-\pi-x, 0), \\ 0 & \text{otherwise}; \end{cases} \quad \chi_I(x-t) = \begin{cases} 1 & \text{if } t \in (0, \pi+x), \\ 0 & \text{otherwise}. \end{cases} \]
The choice of $x \in (0, \pi)$ ensures that $-\pi - x < 0$ and $\pi + x > \pi$, so, for $t \in (0, \pi]$, $g(x, t) = 0$. Hence
\[ \int_0^\pi g(x, t)D_n(t)dt = 0 \]
for all $n$, and the required result follows from Proposition 7.6.1.

**Exercise 4**

Show by using Proposition 7.6.1 that the Fourier series for $x \mapsto \cos Nx$ converges pointwise to that function everywhere, where $N$ is a fixed positive integer.

**Solution** In this case,
\[ g(x, t) = \cos N(x + t) + \cos N(x - t) - 2 \cos Nx = 2 \cos Nx (\cos Nt - 1) \]
(using the identity of Exercise 1). The necessary and sufficient condition for the Fourier series to converge pointwise to $\cos Nx$ is
\[ \cos Nx \int_{-\pi}^\pi (\cos Nt - 1) D_n(t)dt \to 0 \quad \text{as } n \to \infty, \]
where we have used the fact that $\cos Nt$ and $D_n(t)$ are even functions.

Now
\[ \int_{-\pi}^\pi \cos Nt D_n(t)dt = \pi \text{ for } n \geq N \]
(cf. Exercise 1), and
\[ \int_{-\pi}^\pi D_n(t)dt = \pi, \]
from the definition of $D_n$. Therefore
\[ \cos Nx \int_{-\pi}^\pi (\cos Nt - 1) D_n(t)dt = 0 \]
for $n \geq N$, and the condition of Proposition 7.6.1 is satisfied.

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2 **The trigonometric functions**

In Exercise 1 we showed that the trigonometric functions form an orthogonal set. The main result of this section is that of Theorem 7.6.2, which is that they form a complete set in $L^2(-\pi, \pi)$. As pointed out in the Introduction, this then tells us that they form a Schauder basis for $L^2(-\pi, \pi)$, which guarantees the strong convergence to $f$ of the Fourier series of every $f \in L^2(-\pi, \pi)$. This answers Question A of the Introduction affirmatively for every function in $L^2(-\pi, \pi)$.

The proof proceeds as follows. We first establish a useful criterion for the pointwise convergence of a Fourier series, the Localization Principle. (This Principle is also applied in the next section to the problem of pointwise convergence.) We then use the Localization Principle to show that the Fourier series of a step function $\phi$ converges pointwise to the step function (except at the finite number of points of discontinuity. We now use two results about strong convergence from Unit 12 to show that the Fourier series for $\phi$ converges strongly to some element of $L^2(-\pi, \pi)$. 

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The first result is Bessel’s Inequality, Theorem T.12.2.2(a). Since we established in Exercise 1 that the normalized trigonometric functions \( \{e_1, e_2, \ldots \} \) form a countable orthonormal set, Bessel’s Inequality tells us that

\[
\sum_{i=1}^{\infty} a_i^2 \leq \|\phi\|^2, \quad \text{where } a_i = (\phi \cdot e_i).
\]

Then the second result, Theorem T.12.2.3(2) tells us that the Fourier series for \( \phi \), \( \sum_{i=1}^{\infty} a_i e_i \) converges. This second result depends crucially on the fact that \( L^2(-\pi, \pi) \) is complete, and is, therefore, a Hilbert space.

Amalgamating these two results we have the following.

The Fourier series of any step function \( \phi \) in \( L^2 \) with respect to an orthonormal set (not necessarily complete) converges to an element of \( L^2 \).

In the reading passage, reference is made to Theorem 7.5.3 (Weir: page 194) which we have not asked you to read. The preceding result may be substituted for that reference.

We now combine the results about \textit{pointwise} convergence to a step function \( \phi \), and \textit{strong} convergence to some element of \( L^2 \), to show \textit{strong} convergence to \( \phi \). It follows that we may approximate \( \phi \) \textit{in norm} by a trigonometric polynomial \( s_n \). As we can also approximate any \( f \in L^2 \) \textit{in norm} by a step function \( \phi \) (Unit 12, Lemma T.12.3(2)) this shows that we may approximate \( f \) in norm by \( s_n \). This is the theorem we sought to prove.

---

**Notes**

1. **Proposition 7.6.2 (The Localization Principle).** The surprising aspect of this Principle is that any \( r \in (0, \pi] \) will do. This is clear from the proof of the Proposition.

2. **Page 206, line 14.** In order to apply the Riemann–Lebesgue Lemma, we need to know that

\[ g(x,t)(2\sin t/2)^{-1} \]

is integrable over the interval \([r, \pi]\). This follows because:

(a) \( g(x,t) \) is integrable, and therefore measurable;

(b) \( (2\sin t/2)^{-1} \) is continuous on \([r, \pi]\) and bounded by \((2\sin r/2)^{-1}\), and so integrable on \([r, \pi]\) (by Weir: page 47, Theorem 3.3.1), and therefore also measurable;

(c) \( g(x,t)(2\sin t/2)^{-1} \) is the product of two measurable functions, and so measurable (Weir: page 131, Corollary);

(d) finally,

\[ |g(x,t)(2\sin t/2)^{-1}| \leq (2\sin r/2)^{-1}|g(x,t)|, \]

and as the right-hand side is integrable over \([r, \pi]\), \( g(x,t)(2\sin t/2)^{-1} \) is integrable over \([r, \pi]\) by Weir: page 122, Proposition 6.1.1.

3. **Page 207, line 18, Gibbs’ Phenomenon.** We illustrate Gibbs’ Phenomenon in SAQ 10.

4. **Page 207, Theorem 7.6.2.** We outlined the strategy of this proof in our introduction to this reading passage, where we also noted that the reference to Theorem 7.5.3 should be replaced by Theorems T.12.2.2(a) and T.12.2.3(2) of Unit 12. Theorem T.2.3 was covered in Section 2 of Unit 11, and the relevant section of Theorem 7.5.4 (concerning the step function approximation to \( f \) in \( L^2 \)) is Lemma T.12.3(2) of Unit 12.
Exercise 5

Weir: page 214, Exercise 1.

Solution We illustrate the proof for the case when \( f \) is increasing. Then, according to the definition on Weir: page 49, line 2,

\[
f(x + 0) = \inf \{ f(x + t) : t > 0 \}.
\]

Now the limit of \( f(x + t) \) as \( t \) tends to 0 through positive values is \( k \) if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( 0 < t < \delta \),

\[
|f(x + t) - k| < \epsilon.
\]

From the properties of the infimum of a set of real numbers, given any \( \epsilon > 0 \), there is an element of the set differing from the infimum by less than \( \epsilon \).

In our case, if \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
f(x + \delta) - f(x + 0) < \epsilon.
\]

Since \( f \) is monotone, this means that whenever \( 0 < t < \delta \),

\[
|f(x + t) - f(x + 0)| < \epsilon.
\]

This shows that the limit \( k \) exists, and is equal to \( f(x + 0) \).

The cases for \( f(x - 0) \), and decreasing functions, are treated similarly.

Exercise 6

Use the result of the preceding exercise to show that \( f(x + 0) \) and \( f(x - 0) \) exist for a function \( f \) of bounded variation, as defined in the Introduction.

Solution If \( f \) has bounded variation on \([a, b]\), then \( f = f_1 - f_2 \) on \([a, b]\), where \( f_1 \) and \( f_2 \) are increasing on \([a, b]\). From Exercise 5, the limits \( f_1(x + 0) \) and \( f_2(x + 0) \) exist; therefore the limit

\[
f(x + 0) = \lim_{t \to 0^+} f(x + t)
\]

exists. A similar argument shows that \( f(x - 0) \) also exists.

Exercise 7

Suppose that the Fourier series of an element \( f \) of \( L^2(-\pi, \pi) \) converges pointwise to a function \( g \) almost everywhere in \((-\pi, \pi)\). Show that \( g = f \) a.e. in \((-\pi, \pi)\) and that \( g \in L^1(-\pi, \pi) \).

Solution As a consequence of Theorem 7.6.2, the Fourier series of any \( f \in L^2(-\pi, \pi) \) converges strongly to \( f \). By Theorem 7.2.3, a sub-series of the Fourier series converges pointwise to \( f \) almost everywhere. This shows that \( g = f \) a.e. in \((-\pi, \pi)\). Since

\[
L^2(-\pi, \pi) \subseteq L^1(-\pi, \pi),
\]

and \( f \in L^2(-\pi, \pi) \), we have \( g \in L^1(-\pi, \pi) \).
3 Pointwise convergence

We now turn to Question B of the Introduction:

Does the Fourier series for \( f \) converge to \( f \) pointwise?

In this section we prove Jordan's Theorem (Theorem 7.6.3), restated below.

If the function \( f \in L^1(-\pi, \pi) \) has period \( 2\pi \), and is of bounded variation on an interval \([x - r, x + r]\), where \( 0 < r < \pi \), then its Fourier series converges pointwise to \( \frac{1}{2}(f(x + 0) + f(x - 0)) \).

As functions of bounded variation encompass most of the functions met in practice (see SAQs 1–6), this theorem gives a valuable sufficient criterion for pointwise convergence.

The strategy of the proof, which we now outline, is very straightforward. It is the technical details which are rather irksome. However, once you appreciate the overall strategy, these technical details can be seen to fall into place — and are thereby more strongly motivated.

We are told that \( f \in L^1(-\pi, \pi) \) has bounded variation on the interval \([x - r, x + r]\), where \( 0 < r \leq \pi \), and we are required to prove that the Fourier series of \( f \) converges at \( x \) to \( \frac{1}{2}(f(x + 0) + f(x - 0)) \).

(a) We know from Proposition 7.6.1 that the sequence of partial sums \( s_n(x) \to s(x) \)

\[ \int_0^r g(x, t)D_n(t) \, dt \to 0, \]

where \( g(x, t) = f(x + t) + f(x - t) - 2s(x) \).

Putting

\[ s(x) = \frac{1}{2}(f(x + 0) + f(x - 0)), \]

we see that the theorem will be proved if we can show that the above integral tends to 0 as \( n \) tends to infinity.

(b) The Localization Principle (Proposition 7.6.2) tells us that it is sufficient to prove that, for some \( r \in (0, \pi] \),

\[ \int_0^r g(x, t)D_n(t) \, dt \to 0 \]

for convergence to \( s(x) \). This suits our purpose admirably, as we are given information about \( f \) for the range \([x - r, x + r]\) which corresponds to the range \([0, r]\) for \( g(x, t) \).

(c) We can show that the integral \( \int_0^r g(x, t)D_n(t) \, dt \) converges to zero as \( n \) tends to infinity by ensuring, for any \( \varepsilon > 0 \), that

\[ \left| \int_0^r g(x, t)D_n(t) \, dt \right| < \varepsilon \]

for sufficiently large \( n \). Since \( g(x, t) \to 0 \) as \( t \to 0 \), we ought to be able to dominate the integral near the lower limit of integration by the behaviour of \( g(x, t) \). From the form

\[ g(x, t)D_n(t) = g(x, t) \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t}, \]

we can control the integral, away from the possible singularity at \( t = 0 \), by the Riemann–Lebesgue Lemma. Therefore we split the range of integration \([0, r]\) into two parts:

\[ \left| \int_0^r g(x, t)D_n(t) \, dt \right| \leq \left| \int_0^\delta g(x, t)D_n(t) \, dt \right| + \left| \int_\delta^r g(x, t)D_n(t) \, dt \right| \]

by using the standard triangle inequality for real numbers, and treat each part of the range separately.
(d) We treat the lower part of the range,
\[ \int_0^t g(x,t)D_n(t)dt \leq \varepsilon/2, \]
by using a result which is specific to monotone functions, Bonnet’s Mean Value Theorem. This tells us the following.

If \( g_1 \) is a positive increasing function on \([a, b]\) and \( h \) is integrable on \([a, b]\), then \( g_1h \) is integrable on \([a, b]\) and
\[ \int_a^b g_1h = g_1(b)\int_c^b h \]
for some \( c \) in \([a, b]\).

(This result is given as an exercise in Weir: page 134, Exercise 12.) Since our original function \( f \) was of bounded variation, so too is \( g(x, t) \). That is, \( g(x, t) \) is the difference of two positive monotone functions, \( g = g_1 - g_2 \), and so we may apply the Bonnet Mean Value Theorem above, taking \( g_1 \) as one increasing component, \( h \) as the function \( D_n \) (and similarly with the second component).
We thus obtain
\[ \int_0^t g_1(x, t)D_n(t)dt = g_1(x, \delta)\int_\delta^t D_n(t)dt \]
for some \( \delta \in (0, \delta) \) (with a similar result for \( g_2 \)). Since \( g_1(x, \delta) \) may be made as small as we please by choosing \( \delta \) suitably small, we can dominate the integral by \( \varepsilon/2 \) if we can show that
\[ \int_a^b D_n(t)dt \]
is bounded, \( a, b \in [0, \pi] \).

This is the result given as Lemma 7.6.1, and is one of the technical details involved in the proof of Jordan’s Theorem.

(e) The remainder of the integral,
\[ \int_0^\pi g(x, t)D_n(t)dt, \]
may be readily seen to tend to zero as \( n \) goes to infinity from the Riemann–Lebesgue Lemma, using precisely the same argument as in the Localization Principle, Section 2, Note (2).

We therefore obtain
\[ \left| \int_0^\pi g(x, t)D_n(t)dt \right| < \varepsilon/2 \]
for sufficiently large \( n \).

(f) Combining the results of (d) and (e) we have
\[ \left| \int_0^t g(x, t)D_n(t)dt \right| < \varepsilon \]
for sufficiently large \( n \); in view of (c), (b) and (a) this leads to the result sought, namely pointwise convergence of the Fourier series of \( f \) at \( x \) to \( \frac{1}{2} (f(x + 0) + f(x - 0)) \).

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Read Weir: Chapter 7, page 208 to page 211, line 8.

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Notes

1 Page 208, line –5. Weir’s Exercise 17 is our Exercise 8.

2 Page 209, lines 11 and 12. Weir’s Exercise 13 is our Exercise 9. See also Exercise 10 for the derivation of Equation (9).
Define the function $G$ as in the text by

$$G(x) = \int_0^x D_n(t)\,dt.$$ 

Express the area

$$|G(x_k) - G(x_{k-1})| = (-1)^{k-1} \int_{x_{k-1}}^{x_k} D_n(t)\,dt,$$

where $x_k = k\pi/(n + \frac{1}{2})$, as an integral over the interval $[0, \pi]$ by means of a suitable substitution, and use this to show that the area decreases as $k$ increases from 1 to $n + \frac{1}{2}$.

Hence sketch the graph of $G$ from 0 to $\pi$ and show that

$$0 \leq G(x) \leq G(x_1);$$

and that, for $x$, $a$ and $b$ in $[0, \pi]$,

$$|G(b) - G(a)| \leq G(x_1).$$

**Solution** By definition,

$$\int_{x_{k-1}}^{x_k} D_n(t)\,dt = \int_{x_{k-1}}^{x_k} \frac{\sin(n + \frac{1}{2})t}{2\sin \frac{1}{2}t} \,dt.$$ 

The substitution $(n + \frac{1}{2})(t - x_{k-1}) = y$ expresses the integral as

$$\int_0^\pi \frac{\sin(y + (k - 1)\pi)}{\sin ((y + (k - 1)\pi)/(2n + 1))} \frac{dy}{(2n + 1)}.$$ 

Since

$$\sin(y + (k - 1)\pi) = (-1)^{k-1} \sin y,$$

we obtain

$$|G(x_k) - G(x_{k-1})| = \int_0^\pi \frac{\sin y}{\sin ((y + (k - 1)\pi)/(2n + 1))} \frac{dy}{(2n + 1)}.$$ 

For fixed $y \in [0, \pi]$, the denominator increases as $k$ increases from 1 to $n + \frac{1}{2}$.

Therefore the area decreases for the same range of $k$. The curve $G$ has the following form:
From the behaviour of \(|G(x_t) - G(x_{k-1})|\), we see that \(G(x_1)\) is a global maximum of \(G\) in \([0, \pi]\), and in this interval,
\[|G(b) - G(a)| \leq G(x_1).\]

**Exercise 9**


**Solution** See *Weir:* page 272.

**Exercise 10**

Use the results of Exercises 8 and 9 to show that
\[
\left| \int_a^b D_n(t) \, dt \right| \leq M_1
\]
for all \(n = 1, 2, \ldots\) and all \(a, b \in [0, \pi]\), where \(M_1\) is the value \(G(x_1)\) of Exercise 8 for the case \(n = 1\).

**Solution** From the result of Exercise 8, we have
\[
\left| \int_a^b D_n(t) \, dt \right| = |G(b) - G(a)| \leq G(x_1) = \int_0^\pi \frac{\sin y}{\sin (y/(2n+1))} \frac{dy}{(2n+1)}
\]
using the same substitution and notation as in that exercise.

In the notation of Exercise 8,
\[
\left| \int_a^b D_n(t) \, dt \right| \geq \int_0^\pi \frac{v(y)}{v(y/(2n+1))} \, dy.
\]
Since \(v(y)\) is strictly decreasing from 1 to 0 on \([0, \pi]\),
\[
v\left(\frac{y}{2n+1}\right) \geq v\left(\frac{y}{3}\right) \quad \text{for } n = 1, 2, \ldots.
\]
Hence
\[
\int_0^\pi \frac{v(y)}{v(y/(2n+1))} \, dy \leq \int_0^\pi \frac{v(y)}{v(y/3)} \, dy = M_1,
\]
so
\[
\left| \int_a^b D_n(t) \, dt \right| \leq M_1 \quad \text{for } n = 1, 2, \ldots
\]
and all \(a, b \in [0, \pi]\).

**Exercise 11**


**Solution** If \(f\) is even,
\[
b_n = \frac{1}{\pi} \int_{-\pi}^\pi f(t) \sin nt \, dt
\]
\[
= \frac{1}{\pi} \int_{-\pi}^0 f(t) \sin nt \, dt + \frac{1}{\pi} \int_0^\pi f(t) \sin nt \, dt
\]
\[
= -\frac{1}{\pi} \int_0^\pi f(t') \sin nt' \, dt' + \frac{1}{\pi} \int_0^\pi f(t) \sin nt \, dt
\]
\[
= 0 \quad \text{(substituting } t' = -t \text{ in the first integral)}.
\]
Similarly, if \(f\) is odd,
\[a_n = 0 \quad (n = 0, 1, 2, \ldots).\]

**Exercise 12**

*Weir:* page 215, Exercise 3(ii). (Hint: Consider the function \(f : x \rightarrow |x|\).)
Solution: According to the previous exercise, we need an even function on \([-\pi, \pi]\) which is the identity function on \([0, \pi]\); hence the choice of \(f: x \mapsto |x|\). By straightforward evaluation, we obtain \(a_0 = \pi\), and
\[
a_n = \frac{2}{\pi} \int_0^{\pi} t \cos nt \quad (n = 1, 2, \ldots)
\]
\[
= \frac{2}{\pi} \left[ \frac{t \sin nt}{n} \right]_0^{\pi} - \frac{2}{\pi n} \int_0^{\pi} \sin nt \, dt
\]
\[
= -\frac{2}{\pi n^2} [1 - (-1)^n] \quad (n = 1, 2, \ldots).
\]

Therefore, since \(f\) is a function of bounded variation on \([-\pi, \pi]\), being the difference \(g_1 - g_2\) of the two increasing functions
\[
g_1: x \mapsto \begin{cases} x & \text{if } x \in [0, \pi], \\ 0 & \text{otherwise}; \end{cases}
\]
and
\[
g_2: x \mapsto \begin{cases} x & \text{if } x \in [-\pi, 0], \\ 0 & \text{otherwise}, \end{cases}
\]
the result follows from Jordan's Theorem.

4 Summary of the text

In this unit we reviewed the classical Fourier series. We concentrated on answering the two questions on convergence posed in the Introduction.

A Does the Fourier series for \(f\) converge to \(f\) in the sense of a suitable norm?

B Does the Fourier series for \(f\) converge to \(f\) pointwise?

We saw that the appropriate context for discussing Question A was that of the Hilbert space \(L^2(-\pi, \pi)\), and that the convergence was in the sense of the \(L^2\) norm. We proved that the trigonometric functions formed a complete orthonormal set (and therefore a Schauder basis) in \(L^2(-\pi, \pi)\) and this guaranteed convergence to \(f\) in the norm for the Fourier series of every \(f \in L^2(-\pi, \pi)\). This gave a very satisfactory affirmative response to Question A.

The completeness of the set of trigonometric functions was proved using a rather elegant device for expressing the partial sums \(s_n(x)\) of the Fourier series for \(f\), namely the Dirichlet kernels defined by
\[
D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(n + \frac{1}{2})x}{2 \sin \frac{1}{2}x},
\]
thus
\[
s_n(x) = \frac{1}{\pi} \int_0^{\pi} (f(x + t) + f(x - t)) D_n(t) \, dt.
\]
These functions proved their worth in achieving the key result on pointwise convergence, Jordan's Theorem, which answers Question B affirmatively for points about which \(f\) is of bounded variation.

4.1 Postscript on Lebesgue integration

This discussion of Fourier series is an application of the Lebesgue integral to a subject of great practical utility. In particular, the generality of the norm convergence of Fourier series depended on the completeness of the space \(L^2\). This, in
turn, depends on the Monotone Convergence Property of the Lebesgue integral — a property not shared by the Riemann integral, for example. Therefore it is only with the possession of the Lebesgue integral that we can give satisfactory answers to questions such as those concerning the convergence of Fourier series.

In the following subsections references to the set book or this text are preceded by W or T respectively. For example, a reference to page 9 of the set book is indicated by [W9] and to Section 3 of this text by [T3].

4.2 Notation

\[ D_n \quad [W205], \quad [W206] \]
\[ f(x+0), f(x-0) \quad [W207] \]
\[ L^2(\pi, \pi), L^2(-\pi, \pi) \quad [W202], \quad [W203], \quad [T1] \]

4.3 Glossary

Bonnet’s Mean Value Theorem [W134], bounded variation [W68], [T1]
Dirichlet kernels [W205]

4.4 Results

**Theorem 7.6.1 (Riemann-Lebesgue Lemma) [W204]** If \( f \in L^1(\mathbb{R}) \), then the integrals
\[ \int_{-\infty}^{\infty} f(t) \cos kt \, dt \quad \text{and} \quad \int_{-\infty}^{\infty} f(t) \sin kt \, dt \]
both exist and converge to 0 as \( k \to \infty \).

**Proposition 7.6.1 [W205]** Let \( f \in L^1(-\pi, \pi) \) have period \( 2\pi \). Then
\[ s_n(x) \to s(x) \]
if and only if
\[ \int_{0}^{\pi} g(x, t)D_n(t) \, dt \to 0, \]
where
\[ g(x, t) = f(x + t) + f(x - t) - 2s(x). \]

**Proposition 7.6.2 (The Localization Principle) [W206]** Let \( f \in L^1(-\pi, \pi) \) have period \( 2\pi \). Then
\[ s_n(x) \to s(x) \]
if and only if, for some \( r \in (0, \pi] \),
\[ \int_{0}^{\pi} g(x, t)D_n(t) \, dt \to 0. \]

**Proposition 7.6.3 [W206]** Let \( \phi \) be a step function and \( f \) the function of period \( 2\pi \) which coincides with \( \phi \) on \( (-\pi, \pi) \). Then the Fourier series of \( f \) at any point \( x \) converges to the sum
\[ \frac{1}{2}(f(x+0) + f(x-0)). \]

**Theorem 7.6.2 [W207]** The trigonometric functions \( \cos kx \) for \( k \geq 0 \), and \( \sin kx \) for \( k \geq 1 \) form a complete orthogonal set in \( L^2(-\pi, \pi) \).
Theorem 7.6.3 (Jordan) [W209] Let $f \in L^1(-\pi, \pi)$ have period $2\pi$. If $f$ has bounded variation on an interval $[x-r, x+r]$, where $0 < r \leq \pi$, then the Fourier series of $f$ converges at $x$ to the sum
\[ \frac{1}{2} (f(x+0) + f(x-0)). \]

5 Further reading

Those interested in taking the theory of pointwise convergence of Fourier series just a little further than we have done here can do no better than complete the reading of Weir: Section 7.6, pages 211 to 214. The following book details exclusively with the subject matter of this unit.

Alois Kufner and Jan Kadlec, Fourier Series (Butterworth Group, 1971).

There are many texts on Fourier series, at various levels of rigour. The following book is one which we would strongly recommend.


It has a wealth of interesting and non-trivial applications, and it proves all the basic theorems using a very understandable style.

6 Self-Assessment Questions

6.1 Functions of bounded variation

Definition: The function $f : \mathbb{R} \rightarrow \mathbb{R}$ has **bounded variation** on $[a, b]$ if $f$ is the difference of two functions which are increasing on $[a, b]$.

SAQ 1

If $f$ has bounded variation on $[a, b]$, show that it also has bounded variation on the sub-interval $[a_1, b_1]$ of $[a, b]$.

SAQ 2

If $f$ has bounded variation on $[a, b]$ and $[b, c]$, show that it has bounded variation on $[a, c]$.

SAQ 3

Let $g \in L^1(a, b)$ and define the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ by
\[ f(x) = \int_a^x g. \]
Show that $f$ has bounded variation on $[a, b]$.

SAQ 4

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be **smooth** on $[a, b]$ if its derived function $f'$ is continuous on $[a, b]$. Show that a smooth function $f$ has bounded variation on $[a, b]$. (Hint: Use the result of SAQ 3 and the Fundamental Theorem of the Calculus, on Weir: page 57.)
The function $f$ is continuous and piecewise smooth on $[-\pi, \pi]$; that is, we can partition $[-\pi, \pi]$ into a finite number of closed intervals $[-\pi, a_1], [a_1, a_2], \ldots, [a_n, \pi]$ such that the derived function $f'$ is continuous on $[-\pi, a_1], [a_1, a_2], \ldots$. Show that the Fourier series of $f$ converges to $f(x)$ at every point $x$ in $(-\pi, \pi)$.

6.2 General theory

SAQ 6
The abstract form of Parseval’s equation (referred to in Weir: page 211, line 5 as Bessel’s equation) is as follows.

If $\{e_1, e_2, \ldots\}$ is an orthonormal Schauder basis for a Euclidean space $E$, then for each $x \in E$,

$$||x||^2 = \sum_{i=1}^{\infty} (x \cdot e_i)^2$$

(see Unit 12, Theorem T.12.2.2(b)).

Express Parseval’s equation in terms of a function $f$ and its Fourier coefficients.

SAQ 7

The trigonometric series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \sin nx$$

can be shown to converge for every $x \in [-\pi, \pi]$. Show that it is not the Fourier series of any function $f \in L^2(-\pi, \pi)$. (Hint: Use SAQ 6.)

6.3 Specific Fourier series

SAQ 8


SAQ 9
Show that $\sum n^{-4} = \pi^4/90$. (Hint: Use SAQ 8 and SAQ 6.)

SAQ 10


Solutions to Self-Assessment Questions

Solution 1

If $f = g_1 - g_2$, where $g_1$ and $g_2$ are increasing on $[a, b]$, then $g_1$ and $g_2$ are also increasing on $[a_1, b_1]$.

Solution 2

Let $f = g_1 - g_2$ on $[a, b]$ and $f = h_1 - h_2$ on $[b, c]$, where $g_1, g_2$ and $h_1, h_2$ are increasing on their respective domains $[a, b]$ and $[b, c]$. We can ‘fit’ the two intervals together by noting that

$$f(b) = g_1(b) - g_2(b) = h_1(b) - h_2(b),$$

and defining $f = h_1 - h_2$ on $[b, c]$, where

$$h_1(x) = h_1(x) + g_1(b) - h_1(b),$$

$$h_2(x) = h_2(x) + g_2(b) - h_2(b).$$
If we define
\[ f_1 = \begin{cases} g_1 & \text{if } x \in [a, b], \\ h_1 & \text{if } x \in [b, c], \\ 0 & \text{otherwise,} \end{cases} \]
and define \( f_2 \) similarly, then \( f = f_1 - f_2 \), where \( f_1 \) and \( f_2 \) are increasing functions on \([a, b]\).

**Solution 3**
Express \( g \) in the usual way as the difference of two positive integrable functions,
\[ g = g^+ - g^- \]
Then the equation
\[ f(x) = \int_a^x g^+ - \int_a^x g^- \]
expresses \( f \) as the difference of two increasing functions on \([a, b]\).

**Solution 4**
Using the Fundamental Theorem of the Calculus, we may write
\[ f(x) = \int_a^x f' + f(a) \quad \text{for } x \in [a, b]. \]
The result of SAQ 3 now shows that \( f \) has bounded variation. (Clearly the additive constant \( f(a) \) does no harm.)

**Solution 5**
According to the result of SAQ 4, \( f \) has bounded variation on each of the closed intervals \([-\pi, a_1], [a_1, a_2], \ldots, [a_n, \pi]\). Therefore, by SAQ 2, \( f \) has bounded variation on \([-\pi, \pi]\). Since \( f \) is continuous on the closed interval \([-\pi, \pi]\), it is integrable there (Weir: page 48, Corollary 3.3.1), and Jordan's Theorem tells us that the Fourier series of \( f \) converges at each point \( x \in [-\pi, \pi] \) to the sum \( \frac{1}{2} (f(x + 0) + f(x - 0)) \). For \( x \in (-\pi, \pi) \) this sum is \( f(x) \), since \( f \) is continuous at \( x \in (-\pi, \pi) \).

**Solution 6**
In our present context, the Euclidean space \( E \) is \( L^2(-\pi, \pi) \), and the orthonormal Schauder basis comprises the functions
\[ \begin{align*} x \mapsto 1/\sqrt{2\pi} , \\
x \mapsto (1/\sqrt{\pi}) \cos kx , \quad k \geq 1 , \\
x \mapsto (1/\sqrt{\pi}) \sin kx , \quad k \geq 1 . \end{align*} \]
For \( f \in L^2(-\pi, \pi) \),
\[ ||f||^2 = \int_{-\pi}^\pi f^2 , \]
and, for example, if
\[ e_k : x \mapsto (1/\sqrt{\pi}) \cos kx , \]
then
\[ (f \cdot e_k)^2 = \left( \int_{-\pi}^\pi f(x)(1/\sqrt{\pi}) \cos kx \, dx \right)^2 = \pi a_k^2 . \]
Hence Parseval’s equation becomes
\[ \frac{1}{\pi} \int_{-\pi}^\pi f^2 = \frac{1}{2} a_0^2 + \sum_{k=1}^\infty (a_k^2 + b_k^2) . \]

**Solution 7**
From Parseval's equation, the expression \( ||f||^2 \) in terms of the coefficients is
\[ \sum_{n=1}^\infty n^{-1} \] which diverges. Therefore this trigonometric series cannot be the Fourier series of any \( f \in L^2(-\pi, \pi) \).
The function $x \mapsto x^2$ is even, so, by direct calculation,

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} t^2 \cos nt \, dt = 4(-1)^n/n^2 \quad \text{if } n \neq 0,$$

and

$$a_0 = 2\pi^2/3$$

are the only terms which contribute to the Fourier series. Since

$$x^2 = \int_{0}^{\pi} 2t \, dt,$$

we know that our function has bounded variation by SAQ 3. Jordan’s Theorem gives the required pointwise convergence.

**Solution 8**

From SAQ 6, we know that, if $f \in L^2(\pi, \pi)$, then Parseval’s equation, written in terms of the function $f$ and its Fourier coefficients, is

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2 = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \quad (1)$$

In SAQ 8, we showed that the Fourier coefficients for the function $f : x \mapsto x^2$ are

$$a_0 = 2\pi^2/3,$$

$$a_n = 4(-1)^n/n^2 \quad \text{if } n \neq 0,$$

$$b_n = 0.$$

Substituting in Equation 1, we obtain

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \, dx = \frac{1}{2}(2\pi^2/3)^2 + \sum_{n=1}^{\infty} (4/n^2)^2.$$

Hence

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \, dx = 2\pi^4/9 + 16 \sum_{n=1}^{\infty} n^{-4},$$

so

$$\sum_{n=1}^{\infty} n^{-4} = \frac{\pi^4}{16} \left( \frac{2}{5} - \frac{2}{9} \right) = \frac{\pi^4}{90}.$$

**Solution 9**

See Weir: page 273.