THE LEBESGUE INTEGRAL

Unit 2
The Riemann Integral
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Prepared by the Course Team
Set Book


It is essential to have this book; the course is based on it and will not make sense without it.

This unit is not based on the set book.

Bibliography

The following book is referred to quite frequently, and is useful though not essential.


Conventions

Before starting work on this text, please read the *Guide to the Course*.

The set book is referred to as *Weir*, and the above book *Calculus*, by M. Spivak, is referred to as *Spivak*.

Acknowledgement

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Introduction

It might be helpful to you to understand our reasoning in including a unit on Riemann integration in a course on Lebesgue integration. The basic impetus for Lebesgue's construction of the integral bearing his name was a general dissatisfaction with the Riemann integral, a view widely shared by mathematicians at the turn of the century. There were two problems with the Riemann Integral. The first was the limited class of functions that could be integrated. The second was the fact that the convergence of a sequence of functions did not guarantee the convergence of their integrals to the integral of the limit function. The Lebesgue integral deals with these topics in a very satisfactory way, as we shall see presently. Nevertheless, the Riemann integral evidently stands as the reference theory against which an extended integration theory must be measured.

In previous courses you might not have studied the Riemann integral in sufficient depth to provide this background for yourself. Hence this unit has been written to give you the necessary background. To lighten the burden of this course we have decided not to include questions about this unit on the final examination. However, we will be asking you to answer questions about this unit on TMA 01. Some proofs will be seen to be non-assessable. This means that there are no TMA questions about them.

Many of the results in this unit have identifiable counterparts in Lebesgue Integration Theory. As these results will be proved in that context, we have omitted their proofs in this unit. All results in this unit, whether assessed or not, should be borne in mind when studying later units.

We know from M101, The Foundation Course and M203 that

\[ \int_0^1 x^2 \]

represents the area under the parabola \( y = x^2 \) from \( x = 0 \) to \( x = 1 \). And that this integral can be evaluated by evaluating the primitive \( x^3/3 \) of \( x^2 \) at \( x = 1 \) and \( x = 0 \) and taking the difference.

![Figure 1](image)

In symbols,

\[ \int_0^1 x^2 = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}. \]

The alternative approach is to cut the area up into vertical slices with the lines \( x = 1/n, x = 2/n \) etc., and then to approximate these slices from the inside and the outside by rectangles.
The sum of the areas of the larger rectangles is called the upper Riemann sum, and is in this case
\[ \frac{1}{n^3} \sum_{i=1}^{n} \frac{i^2}{n} = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) = \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}. \]

The sum of the areas of the smaller rectangles is called the lower Riemann sum, and is in this case
\[ \frac{1}{n^3} \sum_{i=1}^{n-1} \frac{(i-1)^2}{n} = \frac{1}{n^3} \sum_{i=1}^{n-1} i^2 = \frac{1}{n^3} \cdot \frac{1}{6} n(n-1)(2n-1) = \frac{(n-1)(2n-1)}{6n^2} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}. \]

Both approximating sums converge to $1/3$ as $n$ goes to infinity.

A satisfactory theory of integration has to make this idea of approximating the area under a graph by rectangles precise, and then prove as a theorem the result that areas can be obtained by finding primitives.

One accordingly proceeds as follows in the general case. Note that this definition is worded differently from that used in M203, but is equivalent to it. The form used here is much more convenient for our purposes.

**Definition T.2.1(1)**

(a) A partition $P$ of an interval $[a, b]$ is a finite collection of points of $[a, b]$,

\[ P = \{x_0, x_1, \ldots, x_n\}, \]

where

\[ a = x_0 < x_1 < x_2 < \ldots x_{n-1} < x_n = b. \]

We shall write $I_i = [x_{i-1}, x_i]$ for the $i$th interval defined by the adjacent points $x_{i-1}$ and $x_i$ of the partition.

The length of the $i$th interval is denoted by $\delta x_i = x_i - x_{i-1}$.

The mesh $\|P\| = \max_{1 \leq i \leq n} \{\delta x_i\}$. 

A standard partition is a partition with equally spaced points. Hence all the \( \delta x_i \) are the same, and so \( ||P|| \) is equal to this common length. If there are \( (n+1) \) such points, then \( \delta x_i = (b-a)/n \) and \( ||P|| = (b-a)/n \).

(b) Let \( f \) be bounded on \([a, b]\), let \( P \) be a partition of \([a, b]\), and let

\[
m_i = \inf \limits_{i_i} f \quad \text{and} \quad M_i = \sup \limits_{i_i} f.
\]

The lower Riemann sum of \( f \) is

\[
L(f, P) = \sum_{i=1}^{n} m_i \delta x_i.
\]

The upper Riemann sum of \( f \) is

\[
U(f, P) = \sum_{i=1}^{n} M_i \delta x_i.
\]

(c) The lower (Riemann) integral of \( f \) is

\[
\int_{a}^{b} f = \sup \{ L(f, P) \}.
\]

The upper (Riemann) integral of \( f \) is

\[
\int_{a}^{b} f = \inf \{ U(f, P) \}.
\]

As \( f \) is bounded, the lower and upper integrals both exist. It can be shown — and it seems intuitively correct — that

\[
\int_{a}^{b} f \leq \int_{a}^{b} f.
\]

It is not always the case that these integrals are equal. We say that \( f \) is (Riemann) integrable on \([a, b]\) if

\[
\int_{a}^{b} f = \int_{a}^{b} f;
\]

in this case, the (Riemann) integral of \( f \) on \([a, b]\) is

\[
\int_{a}^{b} f = \int_{a}^{b} f = \int_{a}^{b} f.
\]

The class of integrable functions defined in this way is called the (bounded) Riemann integrable functions. We denote this class of functions by \( BR[a, b] \).

The space \( BR[a, b] \) is large enough to include all the standard functions of analysis. In particular \( BR[a, b] \) contains all the continuous functions and all the monotonic functions. It also includes functions with a finite number of discontinuities. However functions such as Dirichlet’s function,

\[
f(x) = \begin{cases} 
1 & \text{if } x \text{ is rational,} \\
0 & \text{if } x \text{ is irrational,}
\end{cases}
\]

are not in \( BR[a, b] \).

**Exercise 1**

Show that Dirichlet’s function is not Riemann integrable on \([0, 1]\).
Solution Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[0, 1]$. On the subinterval

$I_i = [x_{i-1}, x_i]$

we have

$m_i = \inf_{I_i} f = 0$

and

$M_i = \sup_{I_i} f = 1$.

Then

$L(f, P) = 0 \times \delta x_1 + \cdots + 0 \times \delta x_n = 0,$

and

$U(f, P) = 1 \times \delta x_1 + \cdots + 1 \times \delta x_n = 1,$

since

$\delta x_1 + \cdots + \delta x_n = x_1 - x_0 + x_2 - x_1 + \cdots + x_n - x_{n-1}$

$= x_n - x_0 = 1 - 0 = 1.$

As this is true for all partitions, we conclude that

$\int_0^1 f = 0$

and

$\int_0^1 f = 1,$

so the upper and lower integrals of $f$ are not equal. Hence $f$ is not Riemann integrable.

It is also possible to prove the Fundamental Theorem of Calculus

$\int_a^b f' = f(b) - f(a)$

for functions $f$ with a derivative $f'$ in $BR[a, b]$. This is one of the most important theorems in analysis, and we shall consider it in detail in Section 2.

In the next section we consider a practical criterion for deciding whether or not a function is integrable. In Section 3 we consider singular integrals. These are Riemann integrals where either we wish to integrate over infinite intervals such as $(-\infty, +\infty)$, or where the integrand has singularities of some sort. This then enables us to introduce the Laplace transform of a function, an important tool for solving differential equations. In Section 4 we consider the extension of the integral to two dimensions, called double integrals.

Remark (Non-assessable, down to the end of the Introduction)
The above construction of the upper and lower integrals is due to the French mathematician G. Darboux, in 1875. He showed at the same time that they could be obtained by a limiting process rather than using suprema and infima.
Definition T.2.1(2)

(a) A partition $Q$ is a refinement of a partition $P$ if $P$ is a subset of $Q$, and we write $P \subset Q$. Observe that if $x_{i-1}$ and $x_i$ are adjacent points in $Q$, the interval they define, $[x_{i-1}, x_i]$, is contained in one of the intervals $[y_{j-1}, y_j]$ determined by $P$. It may be that $x_{i-1}$ or $x_i$ is an end-point of $[y_{j-1}, y_j]$, but it may be that $[x_{i-1}, x_i]$ is entirely interior to $[y_{j-1}, y_j]$. If you draw a simple diagram you will quickly see this.

(b) The upper Riemann sums $U(f, P)$ for a bounded function $f$ on $[a, b]$ converge to the number $U(f)$ in the sense of refinement if, given any $\varepsilon > 0$, there is a partition $P(\varepsilon)$ such that

$$|U(f, P) - U(f)| < \varepsilon$$

for all partitions $P$ which are refinements of $P(\varepsilon)$. Similarly, the lower Riemann sums $L(f, Q)$ converge in the sense of refinement to the limit $L(f)$ if, given any $\varepsilon > 0$, there exists a partition $Q(\varepsilon)$ such that

$$|L(f) - L(f, Q)| < \varepsilon$$

for all partitions $Q$ which are refinements of $Q(\varepsilon)$.

(c) We define a second type of limit. Keeping the previous notation, the $U(P, f)$ converge to a limit $U'(f)$ in the sense of mesh if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|U(f, P) - U'(f)| < \varepsilon$$

for all partitions $P$ with mesh $\|P\| < \delta$. In other words, we wish $U(f, P)$ to converge to $U'(f)$ as $\|P\|$ tends to zero. A similar definition holds for the mesh convergence of the lower Riemann sums $L(f, Q)$ to a limit $L'(f)$.

We have two new and somewhat strange notions of convergence now. Darboux's results concerning them are as satisfactory as one could expect.

Theorem T.2.1 If a function $f$ is Riemann integrable in the sense of Definition T.2.1(1), then the four numbers $U(f), L(f), U'(f), L'(f)$ all exist and equal the Riemann integral of $f$.

In the opposite direction, if either pair $U(f), L(f)$ or $U'(f), L'(f)$ exists and the two members of the pair are equal, then $f$ is Riemann integrable.

It follows that if $f \in BR[a, b]$, then

$$\lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n) = \int_a^b f,$$

where $P_n$ is the standard partition of mesh

$$\|P_n\| = (b - a)/n.$$

This reduces convergence calculations to limits of the usual sort in many cases. For this reason, many texts, particularly those stressing applications, 'define' the Riemann integral by this limit.

For conceptual purposes, our original definition is best. For calculational purposes, engineers and physicists usually find Definition T.2.1(2)(c) best.

This ends the non-assessable remark.
1 Integrable functions

In this section we describe a test for integrability known as the Riemann Δ-Criterion.

We then go on to use this test to show that continuous functions are integrable. We also show that integrability persists if we allow a finite number of discontinuities, but that we cannot, in general, allow an infinite number of discontinuities.

We next show that various combinations of integrable functions are integrable. In particular we show that $BR[a, b]$ is a linear space.

1.1 Riemann’s Δ-criterion

Rather than have to go back to the definition each time we want to find out whether a function is integrable or not, it makes life a lot easier if we can apply a simple test of some kind. Various tests for integrability have been devised over the years, but one of the most elegant and useful is the so-called Riemann Δ-Criterion.

If we look again at the partition $P$ of the interval [0, 1] into $N$ equal subintervals, and form the upper and lower approximating sums $U(f, P), L(f, P)$ for the function $f(x) = x^2$, we see that the difference is represented by the sum of the areas of the boxes surrounding the graph of $f$.

Now since we know that the sums $U(f, P), L(f, P)$ both converge to $1/3$ as $n \to \infty$ it follows that

$$\Delta(f, P) = U(f, P) - L(f, P)$$

must go to zero. This means that for any given $\varepsilon > 0$ (however small) it is possible to construct a partition $P$ such that

$$\Delta(f, P) = U(f, P) - L(f, P) < \varepsilon.$$
Explicitly we have (see the Introduction)

\[
\Delta(f, P) = U(f, P) - L(f, P) = \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) - \left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{n}.
\]

So if we wish to make \(\Delta(f, P) < \varepsilon\) all we have to do is choose \(n\) large enough to ensure

\[
\frac{1}{n} < \varepsilon,
\]

or \(n > 1/\varepsilon\). For example, if we were given \(\varepsilon = 10^{-6}\), then we would choose \(n > 10^6\).

In general Riemann’s \(\Delta\)-Criterion gives a necessary and sufficient condition for a bounded function \(f\) to be in \(BR[a, b]\).

**Theorem T.2.1.1 (Riemann’s \(\Delta\)-Criterion)** Let \(f\) be a bounded function on \([a, b]\). A necessary and sufficient condition for \(f\) to be in \(BR[a, b]\) is that, given any \(\varepsilon > 0\), there is a partition \(P\) of \([a, b]\) such that

\[
\Delta(f, P) = U(f, P) - L(f, P) < \varepsilon,
\]

where \(U(f, P), L(f, P)\) are the upper and lower Riemann sums, respectively, and \(P\) will depend on \(\varepsilon\) in general.

### 1.2 Which functions are integrable?

Armed with Riemann’s \(\Delta\)-Criterion we can easily show that most functions one might expect to meet in real life are integrable.

Returning once more to the integral

\[
\int_0^1 x^2,
\]

we showed in Subsection 1.1 that if \(P_n\) is the standard partition of the interval \([0, 1]\) into \(n\) equal subintervals then

\[
\Delta(f, P) = U(f, P) - L(f, P) = \frac{1}{n}.
\]

We used an algebraic argument to prove this there, but the same result can also be obtained geometrically. We know that \(\Delta(f, P)\) is the sum of the areas of the boxes surrounding the graph of \(f\) arising from taking the difference between the the upper and lower approximating areas \(U(f, P), L(f, P)\) for the partition \(P\). Now these boxes all have width \(1/n\), so if we slide them all to the left they pile one on top of another to form a single rectangle of width \(1/n\) and height 1. The total area is \(1/n\), which shows that \(\Delta(f, P) = 1/n\).

![Figure 5](image-url)  
*Figure 5  The case when \(n = 5\).*
The reason why this geometric argument works is that the function \( x^2 \) is increasing over the interval \([0, 1]\). The same argument can clearly be used to show that any increasing function is integrable across any interval. Similarly for any decreasing function. Hence any monotonic function is integrable across any interval.

An alternative geometric argument for demonstrating the integrability of \( x^2 \) over \([0, 1]\) involves sliding the boxes downwards rather than sideways. But we then need the boxes all to have the same height. So what we do is, given \( \varepsilon > 0 \), to choose a partition \( P \) in such a way that each box has height \( \varepsilon \).

![Figure 6](image)

The total area of the boxes is then equal to the area of a single rectangle with height \( \varepsilon \) and width 1. Hence \( \Delta(f, P) < \varepsilon \) for such \( P \).

This second geometric argument can be used to show that any continuous function is integrable over any interval.

**Theorem T.2.1.2** Any continuous function is Riemann integrable; that is, \( C[a, b] \subset BR[a, b] \), where we write \( C[a, b] \) for the set of all continuous functions \( f : [a, b] \rightarrow \mathbb{R} \).

**Proof** Observe first that any continuous function \( f \in C[a, b] \) is certainly bounded. This is the Boundedness Theorem for continuous functions, discussed in M203.

Suppose \( \varepsilon > 0 \) is given. We construct a partition \( P \) of \([a, b]\) as follows. Take \( x_1 \) to be the first point in \([a, b]\) for which \( |f(x_1) - f(x_0)| = \varepsilon \). If there is no such point then take \( x_1 = b \). Next take \( x_2 \) to be the first point to the right of \( x_1 \) such that \( |f(x_2) - f(x_1)| = \varepsilon \). If no such point exists then take \( x_2 = b \). If we continue in this way we obtain an increasing sequence

\[
a < x_1 < x_2 < \ldots < x_i < \ldots \leq b
\]

such that

\[
|f(x_i) - f(x_{i-1})| = \varepsilon
\]

for all \( i \).

This process must terminate, i.e. we must have \( x_n = b \) for some \( n \). This is because we would otherwise obtain an infinite sequence of points \( x_i \) with the above properties. Now this sequence would be increasing and bounded above and so would have to be convergent to some point \( c \) in the interval \([a, b]\). (This is the Monotone Convergence Theorem for sequences, discussed in M203.) But then the continuity of the function \( f \) would require that \( f(x_i) \) converge to \( f(c) \), which is impossible since \( f(x_i) \) always jumps by \( \pm \varepsilon \) at each step.
So we have constructed a partition $P$ for which the boxes surrounding the graph of $f$ all have height $< 2\varepsilon$. Sliding them down to the $x$-axis we find that

$$\Delta(f, P) < 2\varepsilon(b - a).$$

If we now go back and change $\varepsilon$ to $\varepsilon/(b - a)$ we then obtain

$$\Delta(f, P) < \varepsilon.$$

Hence the Riemann $\Delta$-Criterion is satisfied and so we have proved $f \in BR[a, b]$.

### 1.3 Functions with discontinuities

We shall now show that if a bounded function $f$ has a finite number of discontinuities on the interval $[a, b]$ then it is in $BR[a, b]$, but that if it has an infinite number of discontinuities on $[a, b]$ then it may not be in $BR[a, b]$.

An important example which you will see a lot of in this course is a step function. This is a function which has a finite number of discontinuities and is constant between each discontinuity and the next.

![Step function](image)

*Figure 7*

Any step function is integrable over any finite interval.

On the other hand, Dirichlet's function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

is discontinuous everywhere and is not integrable over any interval as we showed in the Introduction.

![Dirichlet's function](image)

*Figure 8*

To demonstrate that functions with a finite number of discontinuities but which are otherwise continuous are integrable we use the Riemann $\Delta$-Criterion.

**Proposition T.2.1.3** Let $f$ be bounded on $[a, b]$ and continuous at all but a finite number of points of $[a, b]$. Then $f$ is Riemann integrable over $[a, b]$. 

12
Proof Suppose that $f$ has discontinuities at $c_1, \ldots, c_k$ in the interval $[a, b]$ and is otherwise continuous on $[a, b]$. Suppose also that $\varepsilon > 0$ is given. We have to construct a partition $P$ of $[a, b]$ for which $\Delta(f, P) < \varepsilon$.

Since $f$ is bounded, by the Boundedness Theorem ($M203$) there must exist $m, M$ such that

$$m \leq f(x) \leq M$$

for all $x \in [a, b]$. It follows that for the $i$th interval $I_i = [x_{i-1}, x_i]$ of any partition $P$ of $[a, b]$ we must have $M_i \leq M$, $m_i \geq m$. (See the Introduction for the notation.) So the contribution to $\Delta(f, P)$ from $I_i$ must be less than or equal to $(M - m)(x_i - x_{i-1})$.

![Diagram showing a function with intervals $I_i$ and bounds $m$ and $M$.](image)

In order to eliminate a messy factor of $2k(M - m) + 1$ at the end of the proof, let us introduce

$$\varepsilon' = \varepsilon/(2k(M - m) + 1).$$

(You will see that this technique is closely related to choosing $\delta = \frac{1}{2}\varepsilon$ or $\delta = \frac{1}{2}\sqrt{\varepsilon}$ in the proofs of limits of functions in the $M203$ units on proofs in analysis.)

Then if we choose a partition $P$ of $[a, b]$ to include the points $c_1 \pm \varepsilon', \ldots, c_k \pm \varepsilon'$, the contribution to $\Delta(f, P)$ from around the intervals around $c_1, \ldots, c_k$ must be less than or equal to $2k\varepsilon'(M - m)$. (If $c_1 = a$ we choose $c_1$ rather than $c_1 - \varepsilon'$; and if $c_k = b$, we choose $c_k$ rather than $c_k + \varepsilon'$.) Since the function is continuous between these intervals we can complete the partition $P$ in such a way that the contribution to $\Delta(f, P)$ from the remainder of $[a, b]$ is less than $\varepsilon'$. (Here we have used the necessity part of the Riemann $\Delta$-Criterion.)

Hence we obtain

$$\Delta(f, P) < 2k\varepsilon'(M - m) + \varepsilon' = (2k(M - m) + 1)\varepsilon' = \varepsilon.$$

That is, $\Delta(f, P) < \varepsilon$, as required.

1.4 New functions for old

Our main concern here is to show that $BR[a, b]$ is a linear space. By this we mean that if $f$ is in $BR[a, b]$ then any constant multiple $af$ of $f$ is also in $BR[a, b]$, and if $f, g$ are in $BR[a, b]$ then their sum $f + g$ is also in $BR[a, b]$.

Other than adding or scaling, we can construct new integrable functions out of old ones by multiplying them together, dividing them (as long as we never divide by zero), or by taking their moduli.

We can also ‘chop up’ an integrable function over an interval $[a, b]$ into integrable functions over subintervals of $[a, b]$. Conversely, we can ‘stick together’ integrable functions over adjacent intervals to make them into integrable functions over larger intervals. These techniques are worth noting as a proposition.
Proposition T.2.1.4

(a) If \( f, g \in BR[a, b] \), then \( f + g \in BR[a, b] \), where we define \( f + g \) as the function whose rule is
\[
(f + g)(x) = f(x) + g(x).
\]

(b) If \( f \in BR[a, b] \) and \( \alpha \in \mathbb{R} \), then \( \alpha f \in BR[a, b] \), where we define \( \alpha f \) to be the function whose rule is
\[
(\alpha f)(x) = \alpha [f(x)].
\]

(c) If \( f, g \in BR[a, b] \), then \( fg \in BR[a, b] \), where \( fg \) is the function whose rule is
\[
(fg)(x) = f(x)g(x).
\]

If \( |g(x)| \geq \delta > 0 \) for some \( \delta \) and all \( x \in [a, b] \), and if we define \( f/g \) to be the function whose rule is
\[
(f/g)(x) = f(x)/g(x),
\]
then
\[ f/g \in BR[a, b]. \]

(d) If \( f \in BR[a, b] \) and \( [c, d] \subseteq [a, b] \), and if we define the restriction of \( f \) to \( [c, d] \) to be the function \( g : [c, d] \rightarrow \mathbb{R} \) given by
\[
g(x) = f(x) \quad \text{if } x \in [c, d],
\]
then \( g \in BR[c, d] \). The function \( g \) is often written as \( f|_{[c,d]} \).

(e) If \( f \in BR[a, b] \) and \( g \in BR[b, c] \), then \( h \in BR[a, c] \), where \( h : [a, c] \rightarrow \mathbb{R} \) is given by
\[
h(x) = \begin{cases} f(x) & \text{if } a \leq x < b, \\ g(x) & \text{if } b < x \leq c. \end{cases}
\]
The value of \( h \) at \( x = b \) is unimportant; other choices also yield functions which are Riemann integrable.

(f) If \( f \in BR[a, b] \), then \( |f| \in BR[a, b] \), where \( |f| : [a, b] \rightarrow \mathbb{R} \) is given by
\[
|f|(x) = |f(x)| \quad \text{for all } x \in [a, b].
\]

Proof. To demonstrate that a constant multiple \( \alpha f \) of an integrable function \( f \) is also integrable we shall consider first the case \( \alpha > 0 \). In this case the graph of \( \alpha f \) is obtained from the graph of \( f \) by means of a vertical scaling by the factor \( \alpha \). (This manipulation of graphs was discussed in the Foundation Course.) So the upper and lower approximating Riemann sums for any partition \( P \) are also scaled by \( \alpha \). Therefore we have
\[
U(\alpha f, P) = \alpha U(f, P),
\]
\[
L(\alpha f, P) = \alpha L(f, P).
\]
Hence
\[
\int_a^b \alpha f = \alpha \int_a^b f = \alpha \int_a^b f = \int_a^b \alpha f.
\]

Next we consider the case \( \alpha = -1 \). The graph of \(-f\) is obtained from the graph of \( f \) by reflecting in the horizontal axis. Therefore for any partition \( P \) we have
\[
U(-f, P) = -L(f, P),
\]
\[
L(-f, P) = -U(f, P).
\]
Hence
\[
\int_a^b (-f) = - \int_a^b f = - \int_a^b f = \int_a^b (-f).
\]
Finally, the case $\alpha < 0$ can be covered by combining the previous two cases. (You should write out the details of this.)

To demonstrate that the sum $f + g$ of two integrable functions $f, g$ is integrable we cannot use a geometric argument because there is no convenient way of representing $f + g$ geometrically in terms of $f$ and $g$. We therefore have to resort to an algebraic approach via the Riemann $\Delta$-Criterion.

The crucial point of the proof is that if $I_i = [x_{i-1}, x_i]$ is the $i$th subinterval of any partition $P$, then

$$M_i(f + g) \leq M_i(f) + M_i(g)$$

and

$$m_i(f + g) \geq m_i(f) + m_i(g).$$

To establish these $m$ inequalities requires more work than we think appropriate here. You can find the details in Spivak, *Supplement to Calculus*, Chapter 13, problem 8.

Granting this, it follows that

$$M_i(f + g) - m_i(f + g) \leq (M_i(f) + M_i(g)) - (m_i(f) + m_i(g))$$

$$= (M_i(f) - m_i(f)) + (M_i(g) - m_i(g)).$$

Therefore, if we multiply by $x_i - x_{i-1}$ and sum over $i$, we obtain

$$\Delta(f + g, P) \leq \Delta(f, P) + \Delta(g, P).$$

Hence if $\varepsilon > 0$ is given, we can find a partition $P$ such that

$$\Delta(f + g, P) < \varepsilon$$

by making sure that $\Delta(f, P)$ and $\Delta(g, P)$ are both less than $\varepsilon/2$.

We shall not give proofs that the other combinations of integrable functions mentioned above are integrable. Proofs can be constructed on similar lines to the proof we have just given for a sum. Any text introducing the Riemann integral ought to have further details.
2 The Fundamental Theorem of Calculus

Having constructed a definition of the integral by referring it to the area under a graph we are now obliged to prove as a theorem that integration and differentiation are inverse processes. There are actually two theorems involved here. One says that if you integrate a derivative you get back the function you first thought of (in some sense.) The other says that if you differentiate an integral (in some sense) you again get back the function you first thought of.

Consider the example

\[ \int_0^1 x^2 = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3}. \]

We are able to evaluate the integral by observing that the integrand \( x^2 \) is the derivative of \( \frac{x^3}{3} \). The value of the integral is obtained by taking the difference between the values of \( \frac{x^3}{3} \) at the end points of the interval \([0, 1]\). More generally if \( f : [a, b] \rightarrow \mathbb{R} \) has a derivative \( f' \in BR[a, b] \), we have

\[ \int_a^b f' = f(b) - f(a). \]

This is the content of Theorem T.2.2.3.

As an illustration of the second theorem, T.2.2.4, consider

\[ \int_0^x t^2 = \left[ \frac{t^3}{3} \right]_0^x = \frac{x^3}{3}. \]

The integral

\[ \int_0^x t^2, \]

regarded as a function of its upper limit of integration \( x \), gives rise to the primitive \( \frac{x^3}{3} \) of the integrand \( t^2 \). More generally if, for given \( f(t) \), we define \( F(x) \) to be

\[ F(x) = \int_a^x f, \]

then \( F'(x) = f(x) \).

We shall show in this section that Riemann’s definition of the integral is adequate to prove both these theorems under the assumption that \( f' \in BR[a, b] \) for the former and \( f \in C[a, b] \) for the latter.

Before we can prove these theorems we need to develop some integration theory: in particular, combination rules and inequalities for integrals. After having proved the theorems, we shall consider applications to the evaluation of integrals and solution of differential equations.

There is a point of notation we have glossed over so far, one which usually is rather troublesome to students. All our integrals have been written without ‘differentials’. Thus we write \( \int_a^b f \), even \( \int_a^x f \). For purposes of calculation, notation such as

\[ \int_a^x f(t)dt \]

is often convenient, especially as regards changing variables (more formally called ‘transformation theory’). We will use whichever notation is most useful at the time, and so may you. Note that whilst we have considered the limit of partitions as the mesh approaches zero, we have nowhere defined any quantity like \( dx \).
2.1 Combination rules for integrals

Unlike differentiation where there is a rule for differentiating almost any combination of functions you can think of, integration theory has very few combination rules. In fact the only combinations of functions that can be integrated easily are sums and constant multiples. For other combinations such as products and moduli we have to be content with inequalities. We shall discuss this further in Subsection 2.2 below. Note that here we are talking about evaluating integrals. We already know that all these combinations preserve integrability.

When it comes to integrating the same function over combinations of different intervals the situation is more satisfactory. There are rules available for integrating over adjacent intervals.

**Proposition T.2.2.1** Using the definitions given in Proposition T.2.1.4, the following Combination Rules hold for Riemann integrals.

(a) If \( f, g \in BR[a, b] \), then

\[
\int_a^b (f + g) = \int_a^b f + \int_a^b g.
\]

(b) If \( f \in BR[a, b] \) and \( \alpha \in \mathbb{R} \), then

\[
\int_a^b (\alpha f) = \alpha \int_a^b f.
\]

(c) If \( c \) is an intermediate point of \([a, b]\), that is, \( a \leq c \leq b \), and \( f \in BR[a, b] \), then

\[
\int_a^b f = \int_a^c f + \int_c^b f.
\]

We interpret

\[
\int_c^c f = 0,
\]

as a convention.

(d) It is consistent with the other definitions and results to define

\[
\int_b^a f = -\int_a^b f
\]

for \( a \leq b \) and \( f \in BR[a, b] \).

(e) If \( x, y, z \) are any three points in any order in \([a, b]\), then for any \( f \in BR[a, b] \),

\[
\int_x^y f + \int_y^z f = \int_x^z f.
\]

![Figure 10](image)

**Proof** The work we did in Subsection 1.4 implies rule (b) for \( \alpha f \). For rule (a), we saw in Subsection 1.4 that \( f + g \in BR[a, b] \), but the proof given there does not enable us to conclude the integration rule immediately.

However we can say from our work there that for any partition \( P \) of \([a, b]\),

\[
M_i(f + g) \leq M_i(f) + M_i(g), \quad m_i(f + g) \geq m_i(f) + m_i(g).
\]
Therefore, multiplying by \( x_i - x_{i-1} \) and summing over \( i \), we have

\[
U(f + g, P) \leq U(f, P) + U(g, P), \\
L(f + g, P) \geq L(f, P) + L(g, P).
\]

It follows that

\[
L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P).
\]

Now, let \( P = P_n \) be the standard partition of \([a, b]\) into \( n \) equal subintervals (see Definition T.2.1(1) and Theorem T.2.1), and if we let \( n \to \infty \), then the outer members of this chain of inequalities both approach

\[
\int_a^b f + \int_a^b g.
\]

Therefore by the Squeeze Rule for sequences (see M203) the inner two members also converge to this limit. But we know that the inner two members both converge to

\[
\int_a^b (f + g).
\]

Hence we must have the equality

\[
\int_a^b (f + g) = \int_a^b f + \int_a^b g,
\]

as required.

As in Proposition T.2.1.4 we shall omit proofs of the rules for integrating the same function over different intervals. They are similar to the one we have just given.

**Exercise 2**

Evaluate the integral

\[
\int_{-1}^1 |x|
\]

by writing it as

\[
\int_{-1}^0 (-x) + \int_0^1 x.
\]

**Solution** Using Proposition T.2.2.1(c), we have

\[
\int_{-1}^0 (-x) = -\int_{-1}^0 x = -\left[\frac{1}{2}x^2\right]_{-1}^0 = \frac{1}{2},
\]

\[
\int_0^1 x = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2},
\]

and so

\[
\int_{-1}^1 |x| = 1.
\]
2.2 Inequalities for integrals

The Riemann integral may be interpreted as the area under the graph of the integrand, at least for well-behaved functions. We can then expect that if a function never takes negative values, its integral will be positive. This is the basic inequality for Riemann integration theory, and is used to derive a number of results.

**Proposition T.2.2.2** If \( f \in BR[a, b] \) and if \( f(x) \geq 0 \) for all \( x \in [a, b] \), then

\[
\int_a^b f \geq 0.
\]

**Proof** This result follows from the fact that here \( L(f, P) \) and \( U(f, P) \) are both greater than or equal to 0 for any partition \( P \) of \([a, b]\). We omit a formal proof.

**Corollary T.2.2.2** If \( f, g \in BR[a, b] \) and if \( f(x) \leq g(x) \) for all \( x \in [a, b] \) then

\[
\int_a^b f \leq \int_a^b g.
\]

**Proof** We simply observe that for all \( x \in [a, b] \)

\[
h(x) = g(x) - f(x) \geq 0,
\]

and therefore

\[
\int_a^b h = \int_a^b g - \int_a^b f \geq 0.
\]

This second result can be expressed verbally by saying that it is legitimate to 'integrate an inequality'.

**Exercise 3**
Show that \( e \geq 5/2 \) by integrating the inequality

\[
ex \geq 1 + x
\]
over the interval \([0, 1]\).

**Solution** Using the corollary we have

\[
\int_0^1 e^x \geq \int_0^1 (1 + x).
\]

Each of these integrals may be evaluated:

\[
\int_0^1 e^x = [e^x]_0^1 = e - 1,
\]

and

\[
\int_0^1 (1 + x) = \left[ x + \frac{x^2}{2} \right]_0^1 = \frac{3}{2}.
\]

Therefore we have the inequality

\[
e - 1 \geq \frac{3}{2},
\]

and hence \( e \geq \frac{5}{2} \) as required.

**Exercise 4**
Show that \( 2 \leq \pi \leq 4 \) by integrating the inequalities

\[
\frac{2x}{\pi} \leq \sin x \leq 1
\]
over the interval \([0, \pi/2]\).
The strategy is the same as in the previous Exercise. The Corollary gives us that
\[ \int_0^{\pi/2} \left( \frac{2x}{\pi} \right) \leq \int_0^{\pi/2} \sin x \leq \int_0^{\pi/2} 1. \]

We evaluate each of these integrals in turn:
\[ \int_0^{\pi/2} \left( \frac{2x}{\pi} \right) = \left[ \left( \frac{2}{\pi} \frac{x^2}{2} \right) \right]_0^{\pi/2} = \frac{\pi}{4}; \]
\[ \int_0^{\pi/2} \sin x = \left[ -\cos x \right]_0^{\pi/2} = +\cos(0) = 1; \]
\[ \int_0^{\pi/2} 1 = [x]_0^{\pi/2} = \frac{\pi}{2}. \]

Then
\[ \frac{\pi}{4} \leq 1 \leq \frac{\pi}{2}. \]

The first inequality yields \( \pi \leq 4 \) and the second yields \( \pi \geq 2 \), so \( 2 \leq \pi \leq 4 \).

Another important inequality for integrals is the Triangle Inequality for Integrals, as in M203.

**Theorem T.2.2.2(1) (The Triangle Inequality for Integrals)** For any \( f \in BR[a,b] \),
\[ \left| \int_a^b f \right| \leq \int_a^b |f|. \]

**Proof** By integrating the inequalities\[ -|f(x)| \leq f(x) \leq |f(x)| \]
over \([a,b] \) we obtain\[ \int_a^b -|f| \leq \int_a^b f \leq \int_a^b |f|. \]
This gives\[ -\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|, \]
and hence\[ \left| \int_a^b f \right| \leq \int_a^b |f| \]
as required.

Finally we mention the celebrated *Schwarz's Inequality*. (Independently due, in one form or another, to Cauchy and to the Russian mathematician Bunyakovski as well as to Schwarz.)

**Theorem T.2.2.2(2)** For any two functions \( f, g \) in \( BR[a,b] \),
\[ \left( \int_a^b fg \right)^2 \leq \int_a^b f^2 \int_a^b g^2. \]
Proof Schwarz’s Inequality can be proved by observing that for any real number \( \lambda \) we have

\[(\lambda f(x) + g(x))^2 \geq 0,\]

and therefore

\[0 \leq \int_a^b (\lambda f + g)^2 = \lambda^2 \int_a^b f^2 + 2\lambda \int_a^b fg + \int_a^b g^2 = A\lambda^2 + 2\lambda + C.\]

The only way this quadratic expression in \( \lambda \) can be non-negative for all values of \( \lambda \) is if its graph lies above the horizontal axis, possibly touching it.

![Positive discriminant](image1.png)

![Vanishing discriminant](image2.png)

![Imaginary discriminant](image3.png)

**Figure 11**

So the quadratic equation

\[A\lambda^2 + 2\lambda + C = 0\]

must either have no real roots or coincident real roots. Hence the discriminant must be non-positive,

\[B^2 < 4AC,\]

which gives Schwarz’s Inequality.

**Remark** Various results given above, such as the rules for integrals in Subsection 2.1 and the Schwarz’s Inequality, are meant to apply to Riemann integrals. When we come to Lebesgue integrals we shall have to state and prove them in their pertinent form. As the definition of the integral will be different, so will the proofs be (in principle at least); and in spite of the similarity of the formulas, the meanings also will be different.

**Exercise 5**

Show that \( e^2 \geq 7 \) by applying Schwarz’s Inequality to the integral

\[\int_0^1 xe^x.\]

**Solution** Taking \( f(x) = x \) and \( g(x) = e^x \), Schwarz’s Inequality gives

\[
\left( \int_0^1 xe^x \right)^2 \leq \int_0^1 x^2 \int_0^1 e^{2x} = \left[ \frac{x^3}{3} \right]_0^1 \left[ \frac{e^{2x}}{2} \right]_0^1 = \frac{e^2 - 1}{6}.
\]
Using integration by parts gives
\[ \int_0^1 xe^x = [xe^x]_0^1 - \int_0^1 e^x \]
\[ = e - [e^x]_0^1 = e - (e - 1) = 1. \]

Hence
\[ 1 \leq \frac{e^2 - 1}{6}, \]
which gives \( e^2 \geq 7 \), as required. (This gives the estimate \( e \geq 2.645751 \ldots \), if you are curious.)

**Exercise 6**
Show that \( \pi^4 \geq 96 \) by applying Schwarz's Inequality to the integral
\[ \int_0^{\pi/2} x \sin x, \]
and hence that \( \pi \geq 3.130169 \ldots \).

**Solution** Taking \( f(x) = x \) and \( g(x) = \sin x \), Schwarz's inequality gives
\[ \left( \int_0^{\pi/2} x \sin x \right)^2 \leq \int_0^{\pi/2} x^2 \int_0^{\pi/2} \sin^2 x \]
\[ = \left[ \left( \frac{1}{3} \right) \left( \frac{\pi}{2} \right)^2 \right] \left( \frac{\pi}{4} \right) \]
\[ = \pi^4 / 96. \]

Using integration by parts gives
\[ \int_0^{\pi/2} x \sin x = [\sin x - x \cos x]_0^{\pi/2} = 1, \]
from which we deduce that \( \pi^4 / 96 \geq 1 \).

### 2.3 Integrating the derivative

We demonstrate the second version of the Fundamental Theorem of Calculus in this subsection.

**Theorem T.2.2.3** (The Second Fundamental Theorem of Calculus) If \( f : [a, b] \rightarrow \mathbb{R} \) has a derivative \( f' \) which is Riemann integrable, that is if \( f' \in BR[a,b] \), then
\[ \int_a^b f' = f(b) - f(a). \]

**Proof** If \( P \) is the partition
\[ a = x_0 < x_1 < \ldots < x_i < \ldots < x_n = b \]
of \([a, b]\), we have
\[ f(b) - f(a) = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \]
\[ = \sum_{i=1}^n (x_i - x_{i-1})f'(c_i) \]
for some \( c_i \) between \( x_{i-1} \) and \( x_i \), by the Mean Value Theorem (see M203).
It follows that
\[ L(f', P) \leq f(b) - f(a) \leq U(f', P). \]
So if we let \( P \) be the standard partition of \([a, b]\) into \( n \) equal subintervals and let \( n \to \infty \) we obtain, by the Squeeze Rule (M203),
\[ \int_a^b f' = f(b) - f(a), \]
as required.

As an application of the Second Fundamental Theorem of Calculus in this form we give the derivation of the rules for integration by parts and integration by substitution.

**Proposition T.2.2.3**

(a) Integration by Parts If \( f, g : [a, b] \to \mathbb{R} \) have derivatives \( f', g' \in BR[a, b] \), respectively, then
\[ \int_a^b f'g = [f(x)g(x)]_a^b - \int_a^b fg'. \]

(b) Integration by Substitution If \( f : [a, b] \to \mathbb{R} \) has a derivative, \( f' \in BR[a, b] \); if \( g : [\alpha, \beta] \to \mathbb{R} \) is one-one, has a derivative \( g' \in BR[\alpha, \beta] \), if \( g(\alpha) = a, g(\beta) = b \); and if \( (f' \circ g)g' \in BR[\alpha, \beta] \), then
\[ \int_a^b f' = \int_\alpha^\beta (f' \circ g)g'. \]

In Leibniz notation this reads
\[ \int_a^b f(x)dx = \int_\alpha^\beta f'(g(t))g'(t)dt. \]

**Proof** The requirements \( g(\alpha) = a, g(\beta) = b \) in part (b) ensure that the integrals 'match up'. What is done here is sometimes known as 'changing variables', substituting \( x = g(t) \). To be clear, we have given the formula in both the abstract and Leibniz notations. The complicated integrability condition \( (f' \circ g)g' \in BR[\alpha, \beta] \) holds if, for example, if \( f' \) is continuous in addition to the other requirements. Now we will tackle the proof of part (a).

The function \( f'(x)g(x) + f(x)g'(x) \) is the derivative of \( f(x)g(x) \) and is in \( BR[a, b] \) since \( f', g' \) are, and since \( f, g \) are continuous. (Recall that 'differentiable implies continuous', from M203.)

Hence
\[ \int_a^b (f'(x)g(x) + f(x)g'(x))dx = [f(x)g(x)]_a^b, \]
which gives the result immediately.

To prove part (b), we note that the function \( f'(g(t))g'(t) \) is the derivative (with respect to \( t \)) of the function \( f(g(t)) \) except possibly at points of discontinuity of \( f(g(t)) \) or its derivative. It is the purpose of the stated conditions to ensure that such points do not cause the following to be incorrect:
\[ \int_\alpha^\beta f'(g(t))g'(t)dt = [f(g(t))]_\alpha^\beta \]
\[ = f(g(\beta)) - f(g(\alpha)) \]
\[ = f(b) - f(a) \]
\[ = \int_a^b f'(x)dx. \]

We omit the proof that the conditions are indeed sufficient for this purpose.
Let us note that having shown the validity of Integration by Substitution, we can replace $f'$ by a function $F$ such that all necessary conditions hold, say $F$ is continuous on $[a, b]$. Then it follows that

$$
\int_a^b F = \int_a^b (F \circ g)g'.
$$

The validity of this result rests on the fact that every $F \in C[a, b]$ has a primitive, as we shall show just below. Moreover, the condition $g' \in BR[\alpha, \beta]$ is necessary for this to be true for all $F \in C[a, b]$.

### 2.4 Differentiating the integral

We demonstrate in this subsection the first version of the Fundamental Theorem of Calculus.

**Theorem T.2.2.4** (The First Fundamental Theorem of Calculus) If $f \in C[a, b]$ and if we define the function $F : [a, b] \rightarrow \mathbb{R}$ by the rule

$$
F(x) = \int_a^x f,
$$

then $F$ is differentiable and $F' = f$.

**Proof** If $h > 0$ we have

$$
\frac{F(x + h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt = \frac{1}{h} \int_x^{x+h} (f(t) - f(x))dt,
$$

using the ‘trick’ replacement

$$
\int_x^{x+h} f(x)dt = hf(x).
$$

Therefore

$$
\left| \frac{F(x + h) - F(x)}{h} - f(x) \right| = \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|dt \leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|dt.
$$

Suppose $\varepsilon > 0$ is given. Then by the continuity of $f$ at $x$ we can choose $\delta > 0$ such that

$$
|f(t) - f(x)| < \varepsilon
$$

for all $t$ satisfying $|t - x| < \delta$. Therefore if $0 < h < \delta$ we have

$$
|f(t) - f(x)| < \varepsilon
$$

for all $t \in [x, x + h]$.

It follows that

$$
\left| \frac{F(x + h) - F(x)}{h} - f(x) \right| \leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|dt \leq \frac{1}{h} \int_x^{x+h} \varepsilon dt = \varepsilon h \rightarrow 0
$$

for all $h$ satisfying $0 < h < \delta$. This shows

$$
\lim_{h \rightarrow 0^+} \frac{F(x + h) - F(x)}{h} = f(x).
$$
Similarly we can also show that
\[ \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = f(x). \]
Hence \( F' \) exists and equals \( f \) at all points \( x \in (a, b) \). For the end points we have shown that
\[ F'(a^+) = f(a) \quad \text{and} \quad F'(b^-) = f(b), \]
which completes the proof.

The importance of the Fundamental Theorem of Calculus in this form is that it assures us that any continuous function has a primitive. Therefore, for example, we can say that \( f(x) = e^{-x^2} \) has the primitive
\[ F(x) = \int_0^x e^{-t^2} dt, \]
even though we cannot express \( F \) in terms of elementary functions.

**Remark** (Non-assessed) Another important example in applied mathematics is the solution, using Green’s Functions, of the differential equation
\[ f''(x) + f(x) = g(x), \]
with initial conditions \( f(0) = f'(0) = 0 \). (See MST204.) The solution is
\[ f(x) = \int_0^x (x - t)g(t)dt, \]
which exists for any continuous \( g \).

Expanding the trigonometric function,
\[ f(x) = \sin x \int_0^x \cos t g(t)dt - \cos x \int_0^x \sin t g(t)dt. \]
Then, using the rule for the derivative of a product,
\[ f'(x) = \cos x \int_0^x \cos t g(t)dt + \sin x \cos x g(x) + \sin x \int_0^x \sin t g(t)dt - \cos x \sin x g(x) \]
\[ = \cos x \int_0^x \cos t g(t)dt + \sin x \int_0^x \sin t g(t)dt. \]
Differentiating again,
\[ f''(x) = -\sin x \int_0^x \cos t g(t)dt + (\cos x)^2 g(x) + \cos x \int_0^x \sin t g(t)dt + (\sin x)^2 g(x) \]
\[ = g(x) - f(x), \]
as required.
3 Singular integrals

Up to this point we have only integrated bounded functions over bounded intervals. We now wish to relax both these restrictions.

Singular integrals of the first kind are those which are taken over an unbounded interval. Singular integrals of the second kind are those which have an unbounded integrand.

For example
\[ \int_{0}^{\infty} e^{-x} \, dx \]
is a singular integral of the first kind whilst
\[ \int_{0}^{1} \frac{dx}{\sqrt{x}} \]
is a singular integral of the second kind (since the integrand is unbounded near 0, and is indeed undefined at \( x = 0 \)).

We evaluate singular integrals by regarding them as limits of non-singular integrals, that is, with bounded integrand and bounded interval of integration as considered earlier. In technical terms, we will extend our integral from the class \( BR[a,b] \) to a larger class of functions.

Example 1 To evaluate
\[ \int_{0}^{\infty} e^{-x} \, dx \]
we consider the non-singular integral
\[ \int_{0}^{b} e^{-x} \, dx = [-e^{-x}]_{0}^{b} = 1 - e^{-b}, \]
and take the limit as \( b \to \infty \). We obtain
\[ \int_{0}^{\infty} e^{-x} \, dx = \lim_{b \to \infty} \int_{0}^{b} e^{-x} \, dx = \lim_{b \to \infty} (1 - e^{-b}) = 1. \]

Example 2 To evaluate
\[ \int_{0}^{1} \frac{dx}{\sqrt{x}} \]
we consider the non-singular integral
\[ \int_{a}^{1} \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_{a}^{1} = 2(1 - \sqrt{a}) \quad (a > 0) \]
and take the limit as \( a \to 0+ \). Hence
\[ \int_{0}^{1} \frac{dx}{\sqrt{x}} = \lim_{a \to 0+} \int_{a}^{1} \frac{dx}{\sqrt{x}} = 2. \]
Example 3 It may happen that the limit we want fails to exist. For example, consider

\[ \int_0^{\pi/2} \tan x \, dx. \]

This is a singular integral of the second kind because \( \tan x = \infty \) at \( x = \pi/2 \). For \( a \in (0, \pi/2) \) we have

\[ \int_0^{(\pi/2)-a} \tan x \, dx = [ - \log \cos x ]_0^{(\pi/2)-a} = - \log \cos((\pi/2) - a). \]

If we let \( a \to 0^+ \) here we get \(- \log \cos \pi/2\), but \( \cos \pi/2 = 0 \) and \( \log 0 = -\infty \)!

We deal with this situation by introducing the concept of **convergence**. We say the integrals

\[ \int_0^\infty e^{-x} \, dx \quad \text{and} \quad \int_0^1 \frac{dx}{\sqrt{x}} \]

**converge**, but that the integral

\[ \int_0^{\pi/2} \tan x \, dx \]

**diverges**.

If the integrand \( f(x) \) of the integral

\[ \int_0^\infty f \]

varies in sign we say the integral converges **absolutely** if

\[ \int_0^\infty |f| \]

converges. The Absolute Convergence Test for Integrals says that any absolutely convergent integral must be convergent. This is similar to the fact that absolutely convergent series are convergent. (You met this in M203.) Similar observations apply to singular integrals of the second kind.

An important class of singular integrals are the so-called **Laplace transforms**

\[ F(x) = \int_0^\infty e^{-xt} f(t) \, dt. \]

The above integral transforms the function \( f(t) \) into another function \( F(x) \) called the **Laplace transform** of \( f(t) \). Laplace transforms are very useful for solving differential equations. (See MST204.) We shall now analyse these cases more formally.
3.1 Singular integrals of the first kind

Definition T.2.3.1 We say that the integral
\[ \int_a^\infty f \]
converges if for each \( b > a \) the non-singular integral
\[ \int_a^b f \]
exists and the limit
\[ \lim_{b \to \infty} \int_a^b f \]
exists. The value of the singular integral is then defined to be
\[ \int_a^\infty f = \lim_{b \to \infty} \int_a^b f. \]
In a similar way we define
\[ \int_{-\infty}^b f = \lim_{a \to -\infty} \int_a^b f, \]
when it exists. However, and this is surprising, we can only define
\[ \int_{-\infty}^{+\infty} f = \lim_{b \to \infty} \int_a^b f + \lim_{a \to -\infty} \int_a^b f = \int_0^\infty f + \int_{-\infty}^0 f, \]
if both \( \int_0^\infty f \) and \( \int_{-\infty}^0 f \) exist. The reason is that \( \lim_{b \to \infty} \int_{-\infty}^{+\infty} f \) can exist without both the integrals \( \int_0^\infty f \), \( \int_{-\infty}^0 f \) existing. Any of the three integrals, \( \int_a^\infty f \), \( \int_{-\infty}^\infty f \) or \( \int_{-\infty}^{+\infty} f \), is said to be a singular integral of the first kind, if it exists.

The basic example, \( f(x) = x \), shows that this is a quite common phenomenon:
\[ \lim_{b \to \infty} \int_0^b x = \lim_{b \to \infty} \frac{1}{2} b^2 = +\infty, \]
\[ \lim_{a \to -\infty} \int_a^0 x = \lim_{a \to -\infty} -\frac{1}{2} a^2 = -\infty, \]
\[ \lim_{c \to \infty} \int_{-c}^c x = \lim_{c \to \infty} (0) = 0. \]
This example shows why we choose the definition of \( \int_{-\infty}^{+\infty} f \) as we do: we certainly do not want \( \int_{-\infty}^{+\infty} f \) to converge solely by virtue of the cancellation of positive and negative areas.

The notations \( BR[a, \infty), BR(-\infty, b] \) and \( BR(-\infty, +\infty) \) are used for the sets of functions integrable over \( [a, \infty) \), \( (-\infty, b] \) and \( (-\infty, +\infty) \), respectively.

For integrals with a positive integrand there is a Comparison Test available.

Proposition T.2.3.1 (The Comparison Test) If \( 0 \leq f(x) \leq g(x) \) for all \( x \geq a \), if \( f \in BR[a, b] \) for all \( b \geq a \), and if we know that
\[ \int_a^\infty g \]
converges, then it follows that
\[ \int_a^\infty f \]
must converge too.
Proof Under the stated conditions,
\[ \int_a^b f \leq \int_a^b g \]
for all \( b \geq a \). We note that both these integrals increase with \( b \). The one on the right converges to a finite value as \( b \to \infty \), so the one on the left must be bounded above. (See M203.) Hence the limit
\[ \lim_{b \to \infty} \int_a^b f \]
must exist.

Evidently this is a test of some practical utility.

**Exercise 7**

Show that the integral
\[ \int_1^\infty \frac{dx}{x^2} \]
converges. Deduce that the integral
\[ \int_1^\infty \frac{dx}{1 + x^2} \]
converges, by applying the Comparison Test.

**Solution** For \( b \geq 1 \) we have
\[ \int_1^b \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^b = 1 - \frac{1}{b} \to 1 \]
as \( b \to \infty \). Therefore
\[ \int_1^\infty \frac{dx}{x^2} = 1 \]
converges.

For \( x \geq 1 \) we have
\[ \frac{1}{1 + x^2} \leq \frac{1}{x^2}. \]
Therefore, by the Comparison Test,
\[ \int_1^\infty \frac{dx}{1 + x^2} \]
converges.

Observe that we have not evaluated the integral
\[ \int_1^\infty \frac{dx}{1 + x^2}, \]
though we could have done so by writing
\[ \int_1^b \frac{dx}{1 + x^2} = \left[\tan^{-1} x\right]_1^b \]
\[ = \tan^{-1} b - \tan^{-1} 1 \to \pi/2 - \pi/4 = \pi/4, \]
as \( b \to \infty \). Hence
\[ \int_1^\infty \frac{dx}{1 + x^2} = \pi/4. \]
Use the Comparison Test to show that
\[ \int_1^\infty e^{-x^2} \, dx \]
converges. We suggest that you compare the integrand with \( e^{-x} \). (An alternative method is to use the result of Exercise 7.)

**Solution**

Now, as \( b \to \infty \). For \( x > 1 \),
\[ e^{-x^2} < e^{-x^2} \leq e^{-x} \]
so the integral \( \int_1^\infty e^{-x^2} \, dx \) converges by the Comparison Test, and its value is less than \( e^{-1} \).

The other method suggested hinges on the inequality
\[ 0 < e^{-x^2} \leq (1 + x^2)^{-1}, \quad \text{for all } x \geq 1. \]
From the previous exercise we know that \( (1 + x^2)^{-1} \in BR[1, \infty) \), so
\[ \int_1^\infty e^{-x^2} \, dx \leq \int_1^\infty (1 + x^2)^{-1} \, dx = \pi/4. \]
This is not as good a bound as \( e^{-1} \), but it is always good to consider alternative solutions.

### 3.2 Singular integrals of the second kind

**Definition T.2.3.2**

Given that
\[ \int_{a+\varepsilon}^b f \]
exists as a non-singular integral for every \( \varepsilon > 0 \), where \( f \) is not necessarily bounded on \([a, b]\), we say that
\[ \int_a^b f \]
converges if the limit
\[ \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f \]
exists. If it does, its value is given by the limit
\[ \int_a^b f = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f, \]
and is said to be a **singular integral of the second kind**.

If \( f \) is bounded, then \( \int_a^b f \) exists as a singular integral of the second kind if and only if \( f \in BR[a, b] \), in which case the two notions of integral coincide. In this sense, the definition of a singular integral is a sensible extension of the definition of the Riemann integral.

As in Subsection 3.1 there is a Comparison Test for integrals with positive integrand.
Proposition T.2.3.2  If \( 0 \leq f(x) \leq g(x) \) for all \( x \in [a, b] \), if \( f \in \mathcal{R}[a + \varepsilon, b] \) for all \( \varepsilon > 0 \), and if we know that
\[
\int_a^b g = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b g
\]
converges as a singular integral of the second kind, then
\[
\int_a^b f = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f
\]
converges as a singular integral of the second kind, with
\[
\int_a^b f \leq \int_a^b g.
\]

We omit the proof, which is an application of the basic Comparison Theorem for positive Riemann integrable functions to each interval \([a + \varepsilon, b]\).

Exercise 9

Show that
\[
\int_0^1 \frac{dx}{x}
\]
diverges, and hence deduce that
\[
\int_0^1 \frac{dx}{\sin x}
\]
also diverges.

Solution  For any \( \varepsilon > 0 \) we have
\[
\int_0^1 \frac{dx}{x} = \left[ \log x \right]_0^1 = -\log \varepsilon \to \infty
\]
as \( \varepsilon \to 0^+ \). The inequality \( \sin x < x \) for \( 0 < x < 1 \) gives
\[
\frac{1}{x} < \frac{1}{\sin x},
\]
so the divergence of
\[
\int_0^1 \frac{dx}{x}
\]
implies that
\[
\int_0^1 \frac{dx}{\sin x}
\]
must diverge also, otherwise the Comparison Test would be violated. (In this case, the negation of the Comparison Test has been used.)

Exercise 10

Show that the integral
\[
\int_1^2 \frac{dx}{\log x}
\]
diverges, by using the inequality
\[
\log x \leq x - 1, \quad \text{for } x \geq 1.
\]
Solution Using the same strategy as in the previous Exercise, we consider
\[ \int_{1+\varepsilon}^{2} \frac{dx}{x-1} = \left[ \log(x-1) \right]_{1+\varepsilon}^{2} = -\log \varepsilon \rightarrow \infty \]
as \( \varepsilon \rightarrow 0^+ \). The inequality \( \log x \leq x - 1 \) for \( x \in [1, 2] \) gives
\[ \frac{1}{x-1} \leq \frac{1}{\log x}, \quad \text{for} \quad 1 < x \leq 2, \]
so the divergence of
\[ \int_{1}^{2} \frac{dx}{x-1} \]
implies the divergence of
\[ \int_{1}^{2} \frac{dx}{\log x}. \]

### 3.3 Absolute convergence

As stated in the preamble to this section, it is useful to introduce singular integrals of the absolute value \( |f(x)| \) of a function \( f(x) \) and test them for convergence.

In both of the following results we assume that \( f \in BR[a, b] \) for all \( b \geq a \). Then \( |f| \in BR[a, b] \) for all \( b \geq a \), and it makes sense to talk about the convergence of \( \int_{a}^{\infty} f \) and \( \int_{a}^{\infty} |f| \). There are examples of functions \( f \) such that \( |f| \in BR[a, b] \) for all \( b \geq a \), such that \( \int_{a}^{\infty} |f| \) converges, yet \( f \notin BR[a, b] \) for any \( b > a \). For example, let \( a = 0 \) and take
\[ f(x) = \begin{cases} e^{-x} & \text{if } x \in \mathbb{Q}, x \geq 0, \\ -e^{-x} & \text{if } x \notin \mathbb{Q}, x \geq 0. \end{cases} \]

This example shows only why we have to require \( f \in BR[a, b] \) for all \( b \geq a \) in all that follows.

**Definition T.2.3.3** For any function \( f : [a, \infty] \rightarrow \mathbb{R} \) such that \( f \in BR[a, b] \) for all \( b \geq a \), we say that the singular integral \( \int_{a}^{\infty} f \) is **absolutely convergent** whenever the integral \( \int_{a}^{\infty} |f| \) converges.

Surprisingly, it is sometimes easier to determine the convergence behaviour of \( \int_{a}^{\infty} |f| \) than of \( \int_{a}^{\infty} f \). When this is so, we have the following result at our disposal.

**Proposition T.2.3.3 (The Absolute Convergence Test)** Let \( f \in BR[a, b] \) for all \( b \geq a \). If the singular integral \( \int_{a}^{\infty} f \) is absolutely convergent, then it is convergent.

**Proof** Suppose we are given that the singular integral \( \int_{a}^{\infty} f \) converges absolutely. We define two associated functions,
\[ g(x) = \max(f(x), 0), \]
\[ h(x) = \max(-f(x), 0). \]

(The max function is discussed in greater detail in later units.)
Then we have $g(x) \geq 0$, $h(x) \geq 0$ and

\[ g(x) + h(x) = |f(x)|, \]
\[ g(x) - h(x) = f(x). \]

It is convenient to note that

\[ g = \frac{1}{2} |f| + \frac{1}{2} f \quad \text{and} \quad h = \frac{1}{2} |f| - \frac{1}{2} f. \]

Because of our condition that $f \in BR[a, b]$ for all $b \geq a$, it follows that the same is true of $g$ and $h$.

It is also clear that $g \leq |f|$ and $h \leq |f|$, and so, by the Comparison Test, the singular integrals $\int_a^\infty g$ and $\int_a^\infty h$ both converge.

It follows that

\[ \int_b^a f = \int_a^b g - \int_a^b h \rightarrow \int_a^\infty g(x)dx - \int_a^\infty h(x)dx \]

as $b \rightarrow \infty$. Hence $\int_a^\infty f$ converges, completing the proof.

**Example 4** As an illustration of the Absolute Convergence Test, we use it to show that the integral

\[ e^{-x} \sin x \, dx \]

converges. We simply observe that

\[ |e^{-x} \sin x| = e^{-x} |\sin x| \leq e^{-x} \]

for all $x \geq 0$, and the integral converges:

\[ \int_0^\infty e^{-x} \, dx = 1. \]

By the Comparison Test, it follows that the integral

\[ \int_0^\infty e^{-x} \sin x \, dx \]

is absolutely convergent. Hence, by the Absolute Convergence Test, it is convergent. □

**Exercise 11**

Find the value of

\[ \int_0^\infty e^{-x} \sin x \, dx, \]

using the method of integration by parts.

**Solution** We have to proceed in several steps. We start by considering the non-singular integral

\[ I_b = \int_0^b e^{-x} \sin x \, dx = \left[-e^{-x} \cos x\right]_0^b - \int_0^b \cos xe^{-x} \, dx. \]

We now integrate by parts for a second time, and find that

\[ I_b = \left[-e^{-x} \cos x\right]_0^b - \left[e^{-x} \sin x\right]_0^b - I_b, \]

Hence

\[ 2I_b = 1 - (\sin b + \cos b)e^{-b}. \]
For the last step we have to take the limit as \( b \to \infty \). We propose that the following is true:

\[
\lim_{b \to \infty} (\sin b + \cos b)e^{-b} = 0.
\]

For, given any \( \varepsilon > 0 \), let \( b_0 = -\log(\varepsilon/2) \). Then for all \( b > b_0 \),

\[
[(\sin b + \cos b)e^{-b}] < 2e^{-b} < \varepsilon.
\]

Then if we let \( b \to \infty \) we find that

\[
\int_0^\infty e^{-x} \sin x \, dx = \lim_{b \to \infty} I_b = \frac{1}{2}.
\]

### 3.4 Conditional convergence

When an integral fails to be absolutely convergent but is nevertheless convergent we say it is *conditionally convergent*. The classic example of a conditionally convergent integral is

\[
\int_1^\infty \frac{\sin x}{x} \, dx.
\]

As a preliminary to demonstrating this, observe that the integrals

\[
\int_1^\infty \frac{\sin x}{x^2} \, dx \quad \text{and} \quad \int_1^\infty \frac{\cos x}{x^2} \, dx
\]

are absolutely convergent, by comparison with the integral

\[
\int_1^\infty \frac{1}{x^2} \, dx.
\]

Now consider the integral

\[
\int_1^b \frac{\sin x}{x} \, dx = \left[ -\frac{\cos x}{x} \right]_1^b - \int_1^b \frac{\cos x}{x^2} \, dx
\]

\[
= \cos 1 - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2} \, dx \to \cos 1 - \int_1^\infty \frac{\cos x}{x^2} \, dx,
\]

as \( b \to \infty \).

This shows that

\[
\int_1^\infty \frac{\sin x}{x} \, dx
\]

is convergent, and that

\[
\int_1^\infty \frac{\sin x}{x} \, dx = \cos 1 - \int_1^\infty \frac{\cos x}{x^2} \, dx.
\]

To show that

\[
\int_1^\infty \frac{\sin x}{x} \, dx
\]

is not absolutely convergent, observe that

\[
\left| \frac{\sin x}{x} \right| \geq \frac{\sin^2 x}{x} = \frac{1}{2} \frac{1 - \cos 2x}{x}.
\]

The last identity yields

\[
\int_1^b \frac{\sin^2 x}{x} \, dx = \frac{1}{2} \int_1^b \frac{dx}{x} - \frac{1}{2} \int_1^b \frac{\cos 2x}{x} \, dx.
\]
Now the integral
\[ \int_1^\infty \frac{dx}{x} \]
diverges, since
\[ \int_1^b \frac{dx}{x} = \log b \rightarrow \infty \]
as \( b \rightarrow \infty \), whilst the integral
\[ \int_1^\infty \frac{\cos 2x}{x} \, dx \]
converges by an argument similar to that used to show that
\[ \int_1^\infty \frac{\sin x}{x} \, dx \]
converges.

It follows that
\[ \int_1^\infty \frac{\sin^2 x}{x} \, dx \]
must diverge, and therefore by the Comparison Test (in its contrapositive form) that
\[ \int_1^\infty \left| \frac{\sin x}{x} \right| \, dx \]
must diverge also. Hence
\[ \int_1^\infty \frac{\sin x}{x} \, dx \]
is not absolutely convergent.

### 3.5 Laplace transforms

Given a function \( f(t) \) defined for \( t \geq 0 \), we can define its Laplace transform
\[ F(x) = \int_0^\infty e^{-xt} f(t) \, dt, \]
powered the integral converges. For example, if \( f(t) \equiv 1 \), then
\[ F(x) = \int_0^\infty e^{-xt} \, dt \]
\[ = \lim_{T \to \infty} \int_0^T e^{-xt} \, dt = \lim_{T \to \infty} \frac{1 - e^{-xT}}{x} = \frac{1}{x} \]
for all \( x > 0 \). So the Laplace transform of \( f(t) \equiv 1 \) converges and has the value
\[ F(x) = 1/x \]
for all \( x > 0 \).

As you may have learnt in previous courses, the Laplace transform is extremely useful in differential equation theory. It is important, therefore, to isolate a condition which is sufficient to guarantee that a function has a Laplace transform in terms of the Riemann integral.

We now introduce the most basic of the possible conditions. It derives from the recognition that \( e^{-xt} \) provides a factor converging to zero as \( t \to \infty \) if \( x > 1 \). Then if \( f(t) \) grows slowly enough compared to \( e^{-xt} \), as \( t \to \infty \) the overall integrand will behave well enough for the integral to exist.
Definition T.2.3.4

(a) For a function \( f : [0, \infty) \to \mathbb{R} \), its Laplace transform is defined to be the function

\[
F(x) = \int_0^\infty e^{-xt} f(t) \, dt
\]

over the domain of \( x \) in \( \mathbb{R} \) for which the integral exists. The integral is to be interpreted as a singular Riemann integral of the first kind. If no such domain of \( x \) exists, we say that \( f \) has no Laplace transform.

(b) We say that a function \( f : [0, \infty) \to \mathbb{R} \) has exponential order with parameter \( \alpha \) if there is a positive constant \( M \) such that

\[
|f(t)| \leq Me^{\alpha t}
\]

for all \( t \geq 0 \). When \( \alpha = 1 \) we often abbreviate this to ‘\( f \) is of exponential order’.

You should be able to guess at the existence theorem for functions of exponential order.

Theorem T.2.3.5 (Existence of Laplace Transforms) If \( f \) has exponential order with parameter \( \alpha \) and \( f \in B[0, T] \) for all \( T \geq 0 \), then its Laplace transform converges absolutely for all \( x > \alpha \). Hence \( F(x) \) exists for all \( x > \alpha \).

Proof For any \( x > \alpha \) we have

\[
|e^{-xt} f(t)| \leq Me^{(\alpha-x)t}.
\]

Observe that

\[
\int_0^T e^{(\alpha-x)t} \, dt = \left[ \frac{e^{(\alpha-x)t}}{\alpha-x} \right]_0^T = \frac{1 - e^{(\alpha-x)T}}{\alpha-x} \rightarrow \frac{1}{x-\alpha}
\]

as \( T \rightarrow \infty \), provided \( \alpha - x < 0 \) to ensure that the exponential term converges to zero. By the Comparison Test,

\[
\int_0^\infty |e^{-xt} f(t)| \, dt \leq \int_0^\infty Me^{(\alpha-x)t} \, dt = \frac{M}{x-\alpha}, \quad x > \alpha.
\]

Thus, the Laplace transform converges absolutely for \( x > \alpha \). By the Absolute Convergence Test, \( F(x) \) is well-defined for \( x > \alpha \).

For example, any polynomial has exponential order with parameter \( \alpha \) for every \( \alpha > 0 \), and hence has a convergent Laplace transform for all \( x > 0 \).

Exercise 12

Show by induction that

\[
\int_0^\infty t^n e^{-xt} \, dt = \frac{n!}{x^{n+1}}
\]

for all \( x > 0 \).
Solution By the remark above, the Laplace transform of \( t^n \) converges absolutely for all \( x > 0 \). The formula is true for \( n = 0 \), as we have shown previously:

\[
\int_0^\infty e^{-xt} \, dt = \lim_{T \to \infty} \left[ \frac{1}{x} e^{-xt} \right]_{t=0}^{t=T} = \lim_{T \to \infty} \left( \frac{1}{x} - \frac{1}{x} e^{-xT} \right) = \frac{1}{x}.
\]

Now assume that

\[
\int_0^\infty t^{n-1} e^{-xt} \, dt = \lim_{T \to \infty} \int_0^T t^{n-1} e^{-xt} \, dt = \frac{(n-1)!}{x^n}.
\]

Using the technique of integration by parts,

\[
\int_0^T t^{n-1} e^{-xt} \, dt = \left[ \frac{t^n}{n} e^{-xt} \right]_0^T + \int_0^T \frac{t^n}{n} x e^{-xt} \, dt
\]

\[
= \frac{T^n}{n} e^{-xT} + \frac{x}{n} \int_0^T t^{n-1} e^{-xt} \, dt.
\]

Rearranging terms and taking the limit \( T \to \infty \) yields

\[
\int_0^\infty t^{n-1} e^{-xt} \, dt = \lim_{T \to \infty} \int_0^T t^{n-1} e^{-xt} \, dt
\]

\[
= \lim_{T \to \infty} \frac{n}{x} \int_0^T t^{n-1} e^{-xt} \, dt - \lim_{T \to \infty} \frac{n}{x} \int_0^T \frac{T^n}{n} e^{-xt} \, dt
\]

\[
= \left( \frac{n}{x} \right) \left( \frac{(n-1)!}{x^n} \right) = \frac{n!}{x^{n+1}}.
\]

This proves that assuming the formula true for \( n - 1 \) implies its truth for \( n \). By the Principle of Mathematical Induction, we conclude that it is true for all \( n = 0, 1, 2, 3, \ldots \) and all \( x > 0 \).

4 Multiple integrals

For any function \( f(x) \) of a single variable \( x \) the equation \( y = f(x) \) represents a curve in the \((x, y)\)-plane and the integral

\[
\int_a^b f
\]

represents the area (positive and negative) under this curve between the limits \( a \) and \( b \).
A function \( F(x, y) \) of two variables \( x, y \) gives rise to a surface \( z = F(x, y) \) in three-dimensional \((x, y, z)\)-space, and if \( \mathcal{R} \) is any region in the \((x, y)\)-plane the integral
\[
\iint_{\mathcal{R}} F(x, y) \, dx \, dy
\]
can be defined and represents the volume \( V(F, \mathcal{R}) \) under this surface standing on the region \( \mathcal{R} \).

Later we shall discover that the evaluation of a double integral is accomplished by first integrating with respect to one variable and subsequently with respect to the other. In general, the limits of integration for the first integral involve functions of the second variable. How such functions are found and the integrals done we shall leave for later. For now let us define the double integral, by as close an analogy with the (single) Riemann integral as possible.

### 4.1 Double Riemann integrals

The basic case is when \( \mathcal{R} \) is a closed rectangle whose sides are defined by the \( x \)- and \( y \)-intervals \([a, b]\) and \([c, d]\) respectively. We then write
\[
\mathcal{R} = [a, b] \times [c, d],
\]
the Cartesian product of these intervals. We can also write
\[
\mathcal{R} = \{(x, y) : a \leq x \leq b, \ c \leq y \leq d\}.
\]

**Definition T.2.4.1**

(a) A partition of \( \mathcal{R} = [a, b] \times [c, d] \) is an ordered pair \((P', P'')\), where \( P' \) is a partition of \([a, b]\) and \( P'' \) is a partition of \([c, d]\).

If we write \( P' = \{x_0, x_1, \ldots, x_n\} \) and \( P'' = \{y_0, y_1, \ldots, y_m\} \), then we have intervals
\[
I'_i = [x_{i-1}, x_i] \quad \text{and} \quad I''_j = [y_{j-1}, y_j]
\]
of \([a, b]\) and \([c, d]\), respectively, as we did in Definition T.2.1(1). Similarly, \((P', P'')\) defines the rectangles
\[
\mathcal{R}_{ij} = I'_i \times I''_j, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,
\]
which are subrectangles of \( \mathcal{R} \).
(b) Let $F : [a, b] \times [c, d] \to \mathbb{R}$ be bounded and let $Q$ be a partition of $[a, b] \times [c, d]$. For each $i, j$ we define the least upper bound and the greatest lower bound of $F$ on the subrectangle $R_{ij}$ by the rules

$M_{ij} = \sup \{F(x, y) : (x, y) \in R_{ij}\}$

and

$m_{ij} = \inf \{F(x, y) : (x, y) \in R_{ij}\}$.

Summing over all the subrectangles we define the upper and lower Riemann sums, respectively:

$U(F, Q) = \sum_{j=0}^{n} \sum_{i=0}^{m} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$,

$L(F, Q) = \sum_{j=0}^{n} \sum_{i=0}^{m} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$.

In Figure 18 we show an $L(F, Q)$ division in a particularly simple case.

(c) The lower Riemann integral of $F$ is

$\int \int_{\mathcal{R}} F = \sup_{Q} \{L(F, Q)\}$

and the upper Riemann integral of $F$ is

$\int \int_{\mathcal{R}} F = \inf_{Q} \{U(F, Q)\}$.

These upper and lower bounds are over all partitions $Q$ of the closed rectangle $\mathcal{R}$.
As $F$ is bounded, the lower and upper Riemann integrals both exist, and (as can be shown)
\[ \iint_{\mathcal{R}} F \leq \iint_{\mathcal{R}} F. \]
We say that $F$ is Riemann integrable over $\mathcal{R}$ if both these integrals are equal, and in this case we say that their common value is the Riemann integral of $F$ over $\mathcal{R}$, and write
\[ \iint_{\mathcal{R}} F = \iint_{\mathcal{R}} F = \iint_{\mathcal{R}} F. \]
The set of all Riemann integrable functions on $\mathcal{R}$ is written as $BR(\mathcal{R})$.

**Remark T.2.4.1** It is possible to develop a theory of integration for double integrals which parallels closely the theory for single integrals. For our purposes it will suffice just to mention some results in an informal manner.

1. If $\mathcal{D}$ is a bounded non-rectangular region it can be enclosed in a closed rectangle $\mathcal{R}$. It does not matter which rectangle $\mathcal{R}$ we consider, only that $\mathcal{D} \subseteq \mathcal{R}$. Suppose $F : \mathcal{D} \to \mathbb{R}$ is a bounded function. Consider the function $G : \mathcal{R} \to \mathbb{R}$ given by
\[ G(x, y) = \begin{cases} F(x, y) & \text{if } (x, y) \in \mathcal{D}, \\ 0 & \text{if } (x, y) \in \mathcal{R} \setminus \mathcal{D}. \end{cases} \]
An example is shown in Figure 19.

![Figure 19](image-url)

We say that $F$ is Riemann integrable on $\mathcal{D}$, $F \in BR(\mathcal{D})$ if and only if $G \in BR(\mathcal{R})$, and define
\[ \iint_{\mathcal{D}} F = \iint_{\mathcal{R}} G. \]
It is elementary that this definition is independent of the choice of enclosing rectangle $\mathcal{R}$.

An interesting sidelight on this occurs if we take $F(x, y) = 1$ for $(x, y) \in \mathcal{D}$ and zero otherwise. Then
\[ \iint_{\mathcal{D}} F = A(\mathcal{D}) \]
is the area of $\mathcal{D}$, if it exists. You may be surprised to learn that there are bounded subsets of the plane which have no 'area' in this sense.
If $D$ is disconnected enough, this is not hard to see. If we take $D$ to consist of those points $(x, y) \in [0,1] \times [0,1]$ for which both $x$ and $y$ are rational, the inner area of $D$ is 0, the outer area is 1. This is the same sort of calculation we did for the Dirichlet function.

Of far more interest are the sets now being studied in chaos/fractal theory. The justly famous Mandelbrot set, for example, is connected, bounded and has no 'Riemann area'. Much as we would like to pursue this topic, to do so would take us too far afield.

2 For unbounded sets in the plane, $D \subset \mathbb{R}^2$, we proceed as for singular integrals of the first kind. Namely, we consider an increasing sequence $D_n$ of bounded regions, $D_n \subset D_{n+1}$, such that as $n$ increases indefinitely, these regions eventually approach $D$. We then compute the integrals $\int_{D_n} F$, and if these converge as $n \to \infty$, the limit is defined to be the integral $\int_D F$.

There is no unique way to approach $D$ by sequences: so we need to see whether or not the limit

$$\lim_{n \to \infty} \int_{D_n} F$$

is independent of the choice of $D_n$. Fortunately, these matters will not concern us in this course, and Lebesgue theory handles such problems far more efficiently.

In like fashion, if $F$ is unbounded on a bounded region $D$, we might be able to find a sequence of regions $D_n \subset D$ such that $D_n \subset D_{n+1}$, the $D_n$ approach $D$, and $F$ is bounded on each $D_n$. Then we can define

$$\int_D F = \lim_{n \to \infty} \int_{D_n} F,$$

provided the limit exists. Similar problems of definition arise, and, similarly we shall not need to consider them. Note that this is the two-dimensional analogue of singular integrals of the second kind. For when we considered

$$\int_{a+\varepsilon}^{b} f,$$

say, our function $f$, whilst unbounded on $[a, b]$, was bounded on $[a + \varepsilon, b]$.

3 Which functions are Riemann integrable over bounded regions $D$? First of all, functions which are bounded and continuous except at a finite number of points. Closer analysis reveals that functions which are bounded may have an infinite number of discontinuities, but both the type of discontinuity and the set of discontinuous points must be carefully controlled. Again, Lebesgue theory handles this problem more efficiently, and we shall be content to consider which functions have infinitely many singularities and are Lebesgue integrable.

4 We have defined the area of a region of the plane in terms of Riemann integrals above. We then pointed out that this was not the most general definition of area: the Lebesgue integral gives a well-defined area to more general planar regions.

We can also define a notion of volume, based on the Riemann integral. Again, the Lebesgue integral will yield a more general notion. Quite simply, if $F : D \to \mathbb{R}$ is Riemann integrable over the planar region $D \subset \mathbb{R}^2$, so $F \in BR(D)$, we interpret

$$\int_D F = V(F, D)$$

as the volume of the three-dimensional region interior to the figure bounded by $D$ (as the 'base') and the values of $F$ (as the 'height'). In the next section we shall consider how to evaluate $V(F, D)$ in certain cases.
4.2 Evaluation of double integrals

The usual method of evaluation of integrals for functions of two variables involves performing successive single integrations of functions of one variable. We begin with an informal discussion of the method for rectangular regions \( \mathcal{R} \), and then we shall state the corresponding theorem formally.

For a function \( F \) on \( \mathcal{R} \) such that \( F \geq 0 \) our objective is to evaluate the volume of \( V(F, \mathcal{R}) \). Consider the subset of \( V(F, \mathcal{R}) \) bounded by the planes \( y = y_{i-1} \) and \( y = y_i \), and having volume \( V_i \), say.

![Diagram of volume V(F,R)](image)

Suppose that the area of the plane section of \( V(F, \mathcal{R}) \) with the plane at height \( y \) is \( f(y) \); then the volume \( V_i \) is approximately \( f(y_i)(y_i - y_{i-1}) \), so that the volume of \( V(F, \mathcal{R}) \) is approximately

\[
\sum_{i=1}^{n} f(y_i)(y_i - y_{i-1}).
\]

It seems very reasonable that we can then have

\[
V(F, \mathcal{R}) = \int_{c}^{d} f(y)dy.
\]

But we can determine \( f(y) \) directly from \( F \); for any value of \( y \in [c, d] \) we have

\[
f(y) = \int_{a}^{b} F(x, y)dx,
\]

so that

\[
\int \int_{\mathcal{R}} F = \int_{c}^{d} \left( \int_{a}^{b} F(x, y)dx \right) dy.
\]

The integral on the right is often called a repeated integral, and it is important to notice that \( y \) is treated as a constant for the first stage of integration, but at the second stage we ‘integrate with respect to \( y \)’. (There is a parallel here with the process of partial differentiation.)

It seems very reasonable that we should also be able to prove that

\[
\int \int_{\mathcal{R}} F = \int_{a}^{b} \left( \int_{c}^{d} F(x, y)dy \right) dx.
\]

Comparing these two expressions for \( V(F, \mathcal{R}) \), it would then follow that the order of integration is reversed. Under certain circumstances (see Theorem and Corollary below) this is indeed the case.
In the following example we shall assume that $F$ is such that for the rectangle $\mathcal{R} = [0, 3] \times [0, 2]$, the equalities

$$
\int \int_{\mathcal{R}} F = \int_0^2 \left( \int_0^3 F(x,y) \, dx \right) \, dy = \int_0^3 \left( \int_0^2 F(x,y) \, dy \right) \, dx
$$

hold.

**Example 5** Evaluate the integral $\int \int_{\mathcal{R}} F$ for the function $F(x,y) = x^2 + y$, and the rectangle $\mathcal{R} = \{(x, y) : 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

We have

$$
\int \int_{\mathcal{R}} (x^2 + y) = \int_0^2 \left( \int_0^3 (x^2 + y) \, dx \right) \, dy
$$

$$
= \int_0^2 \left[ \frac{x^3}{3} + xy \right]_0^3 \, dy
$$

$$
= \int_0^2 \left[ \frac{3^3}{3} - \frac{0^3}{3} + y(3 - 0) \right] \, dy
$$

$$
= \int_0^2 \left[ 9 + 3y \right] \, dy
$$

$$
= \left[ 9y + \frac{3}{2}y^2 \right]_0^3
$$

$$
= 18 + 6 = 24.
$$

Next we can evaluate the repeated integral in reverse order, and so test the assumption that in this case

$$
\int \int_{\mathcal{R}} F = \int_0^3 \left( \int_0^2 F(x,y) \, dy \right) \, dx
$$

$$
= \int_0^3 \left[ \int_0^2 \left( x^2 + y \right) \, dx \right] \, dy
$$

$$
= \int_0^3 \left[ \frac{x^3}{3} + \frac{1}{2}y^2 \right]_0^2 \, dy
$$

$$
= \int_0^3 \left[ x^2(2 - 0) + \left( \frac{2^2}{2} - \frac{0^2}{2} \right) \right] \, dy
$$

$$
= \int_0^3 \left[ 2x^2 + 2 \right] \, dy
$$

$$
= \left[ \frac{2}{3}x^3 + 2x \right]_0^3
$$

$$
= 18 + 6 = 24.
$$

Although the order of integration has made no difference to the final outcome, it is as well to be warned that there can often be a significant difference in the work involved.

Our next step clearly is to prove that, under certain conditions, the technique used in the previous example is valid. We shall now follow common practice, and omit the large brackets in the double integral expression, so that for example

$$
\int_0^d \int_0^b F(x,y) \, dx \, dy \text{ means } \int_0^b \left( \int_0^d F(x,y) \, dx \right) \, dy.
$$
Theorem T.2.4.2 (The Repeated Integral Theorem) If $F$ is integrable on $\mathcal{R} = [a, b] \times [c, d]$ and $\int_a^b F(x, y)dy$ exists for each value of $y \in [c, d]$, then the repeated integral

$$\int_c^d \int_a^b F(x, y)dydx$$

exists, and equals the Riemann integral of $F$ over $\mathcal{R}$:

$$\int \int_{\mathcal{R}} F = \int_c^d \int_a^b F(x, y)dydx.$$

Two useful extensions of the theorem can be deduced when additional conditions on $F$ are included. Note that we have called the first corollary Fubini's theorem. This is not usually done. Fubini proved the corresponding result for Lebesgue integrals. If a distinction is necessary we could call this Fubini's theorem for Riemann integrals.

Corollary T.2.4.2(1) (Fubini's Theorem) If in addition to the conditions of Theorem T.2.4.2 we assume that $\int_a^d F(x, y)dy$ exists for each $x \in [a, b]$ then we have

$$\int \int_{\mathcal{R}} F = \int_c^d \int_a^b F(x, y)dydx = \int_a^b \int_c^d F(x, y)dydx.$$

Corollary T.2.4.2(2) If $F$ is continuous on $\mathcal{R}$, then $\int \int_{\mathcal{R}} F, \int_c^d F(x, y)dy$ for all $x \in [a, b]$, and $\int_a^b F(x, y)dx$ for all $y \in [c, d]$ all exist and so are equal.

Hence, for continuous functions, and in particular for the function $(x, y) \mapsto x^2 + y$ of Example 5, the technique we have used is valid.

Exercise 13

Suppose $\mathcal{R}$ is the unit square $\{(x, y): 0 \leq x \leq 1, \ 0 \leq y \leq 1\}$ and $F$ is the continuous function

$$F(x, y) = x^2 + 2xy + 3y^2$$

defined on $\mathcal{R}$. Evaluate $\int \int_{\mathcal{R}} F$ twice by integrating in the two different orders.

Solution

Now, writing the integral in the form

$$\int \int_{\mathcal{R}} (x^2 + 2xy + 3y^2) = \int_0^1 \int_0^1 (x^2 + 2xy + 3y^2)dydx$$

enables us to integrate with respect to $x$ first, yielding

$$\int_0^1 (x^2 + 2xy + 3y^2)dx = \left[ \frac{1}{3}x^3 + x^2y + \frac{3}{2}y^2 \right]_{x=0}^{x=1}$$

$$= \frac{1}{3} + y + \frac{3}{2}y^2.$$ 

Then integrating this expression with respect to $y$ from 0 to 1, we have

$$\int_0^1 \left( \frac{1}{3} + y + \frac{3}{2}y^2 \right)dy = \left[ \frac{1}{3}y + \frac{1}{2}y^2 + \frac{3}{8}y^3 \right]_{y=0}^{y=1}$$

$$= \frac{1}{3} + \frac{1}{2} + \frac{3}{8} = \frac{11}{8}.$$ 

Hence

$$\int \int_{\mathcal{R}} (x^2 + 2xy + 3y^2) = \frac{11}{8}.$$ 

To integrate in the opposite order, we write

$$\int \int_{\mathcal{R}} (x^2 + 2xy + 3y^2) = \int_0^1 \int_0^1 (x^2 + 2xy + 3y^2)dydx.$$
Then integrating this expression with respect to \( y \) first, we find
\[
\int_0^1 (x^2 + 2xy + 3y^2) \, dy = [x^2y + xy^2 + y^3]_{y=0}^{y=1}
= x^2 + x + 1.
\]
Then integrating with respect to \( x \) from 0 to 1, we obtain the expected result:
\[
\int_0^1 (x^2 + x + 1) \, dx = \left[ \frac{1}{3}x^3 + \frac{1}{2}x^2 + x \right]_{x=0}^{x=1}
= \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}.
\]

For a general region \( D \) the situation is more complicated. Nevertheless, Fubini’s Repeated Integral Theorem still holds, suitably modified. We illustrate this with an example.

In this example we use a standard technique for evaluating integrals, but one you may not be familiar with. For general regions, integrating along planes \( y = \text{constant} \) will require that the limits on the \( x \)-integral be functions of \( y \). For rectangular regions, these functions are simply constants. It would take us too far afield to develop this technique here. If you cannot follow this example, do not be concerned. We shall not assess the technique of integration over general regions.

Suppose that \( D \) is the triangular region \( \{(x, y) : 0 < x < y < 1\} \). Then we have
\[
\int \int_D (x^2 + 2xy + 3y^2) = \int_0^1 \left( \int_0^y (x^2 + 2xy + 3y^2) \, dx \right) \, dy
= \int_0^1 \left( \frac{1}{3}x^3 + x^2y + 3xy^2 \right)_{x=0}^{x=y} \, dy
= \frac{13}{3} \int_0^1 y^3 \, dy
= \left[ \frac{13}{12}y^4 \right]_0^1 = \frac{13}{12}.
\]

Note that in the first integral, the other variable, \( y \), is treated as a constant, and appears in the integration limits.
Integrating in the opposite order we must now write
\[
\int \int_D (x^2 + 2xy + 3y^2) = \int_0^1 \left( \int_x^1 (x^2 + 2xy + 3y^2) dy \right) dx
\]
\[
= \int_0^1 (x^2y + xy^2 + y^3)_{y=x}^1 dx
\]
\[
= \int_0^1 (x^2 + x + 1 - 3x^3) dx
\]
\[
= \left[ \frac{1}{3}x^3 + \frac{1}{2}x^2 + x - \frac{3}{4}x^4 \right]_0^1 = \frac{13}{12}.
\]
(Here \(x\) is first treated as a constant, and appears in the integration limits.)

We do not intend this unit to develop your skills at evaluating Riemann double integrals — commendable though that would be — and so we now turn to the problem of changing variables in integrals.

4.3 Change of variable

In Proposition T.2.2.3 we gave a formula for change of variable in a one-dimensional integral under the name integration by substitution. In slightly different notation it can be written as
\[
\int_{x=a}^{x=b} f(x) dx = \int_{t=\alpha}^{t=\beta} f(x(t)) \frac{dx}{dt} dt,
\]
where \(x\) is replaced by the function \(x(t)\) (called \(g(t)\) previously) of \(t\), \(dx/dt\) is the derivative of this function and \(x(\alpha) = a, x(\beta) = b\).

Example 6 The integral arising from computing the area of a semi-circle of radius \(a\) is (you need not worry where it comes from)
\[
A = \int_{-a}^a \sqrt{a^2 - x^2} dx,
\]
which can be evaluated by substituting \(x = a \sin t\) to obtain
\[
A = \int_{-\pi/2}^{\pi/2} \sqrt{a^2 - a^2 \sin^2 t} \ a \cos t \ dt
\]
\[
= a^2 \int_{-\pi/2}^{\pi/2} \cos^3 t \ dt
\]
\[
= \frac{1}{2}a^2 \int_{-\pi/2}^{\pi/2} (1 + \cos 2t) dt
\]
\[
= \frac{1}{2}a^2 \left[ t + \frac{1}{2} \sin 2t \right]_{-\pi/2}^{\pi/2} = \frac{1}{2} \pi a^2.
\]
Proposition T.2.4.3  (Two-Dimensional Substitution Rule) A change of variables in a two-dimensional integral

\[ \int \int_D f(x, y) dx dy \]

involves replacing \( x, y \) by two functions \( x(t, u), y(t, u) \) of two variables \( t, u \), subject to appropriate conditions. The analogue of \( dx/dt \) in one variable is the so-called Jacobian determinant

\[ \frac{\partial(x, y)}{\partial(t, u)} = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{vmatrix}. \]

The formula is

\[ \int \int_D f(x, y) dx dy = \int \int_S f(x(t, u), y(t, u)) \left| \frac{\partial(x, y)}{\partial(t, u)} \right| dt du, \]

where \( S \) is the region of integration expressed in terms of the new variables \( t, u \).

Example 7  The integral arising from computing the volume of a hemisphere of radius \( a \) is

\[ V = \int \int_D \sqrt{a^2 - x^2 - y^2} \, dx dy, \]

where \( D \) is the circular region \( x^2 + y^2 \leq a^2 \).

![Figure 23](image)

If we use polar co-ordinates \( r, \theta \) we have \( x = r \cos \theta, \quad y = r \sin \theta, \) and the whole area is given by \( \{(r, \theta) : 0 \leq r \leq a, 0 \leq \theta < 2\pi \} \).

![Figure 24](image)

The Jacobian is

\[ \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta r \cos \theta & r \cos^2 \theta + r \sin^2 \theta \end{vmatrix} = r. \]
The new region $S$ of integration is $\{(r, \theta): 0 \leq r \leq a, \ 0 \leq \theta < 2\pi\}$. So we get

\[
V = \int \int_S \sqrt{a^2 - r^2} \, r \, dr \, d\theta \\
= \int_0^{2\pi} d\theta \int_0^a r \sqrt{a^2 - r^2} \, dr \\
= \left( \int_0^{2\pi} d\theta \right) \times \left( \int_0^a r \sqrt{a^2 - r^2} \, dr \right) \\
= 2\pi \left[ \frac{(a^2 - r^2)^{3/2}}{3} \right]_0^a \\
= \frac{2}{3} \pi a^3. \quad \square
\]

### 4.4 Applications

We now show how double integrals can be used to evaluate the singular integrals

\[
\int_0^\infty e^{-x^2} \, dx, \quad \int_0^\infty \frac{\sin x}{x} \, dx.
\]

These integrals are not amenable to the usual methods of evaluation because their integrands have no elementary primitives.

Our first difficulty is that these are singular integrals, as the range of integration is $[0, \infty)$. What we shall do is integrate over $[0, n]$, where $n$ is any integer; and then take the limit as $n \to \infty$. This will provide the justification for techniques, taught in applied subjects, of treating $\infty$ as a number!

To evaluate the first integral we begin by evaluating

\[
\left( \int_0^n e^{-x^2} \, dx \right)^2 = \int_0^n e^{-x^2} \, dx \int_0^n e^{-y^2} \, dy = \int \int_{S(n)} e^{-x^2-y^2},
\]

where $S(n) = [0, n] \times [0, n]$, which is a square in the first quadrant. Note that as $n$ gets large, the square covers more and more of the first quadrant. These equalities are justified by Fubini's Theorem for the square $S(n)$.

The next step is to transform to polar co-ordinates. However, the square is not easily expressed in polar co-ordinates. What we do is inscribe and circumscribe the square in quarter-discs:

\[
D(n) \subset S(n) \subset D(\sqrt{2}n),
\]

with

\[
D(m) = \{(x, y): x, y \geq 0, 0 \leq x^2 + y^2 \leq m^2\} = \{(r, \theta): 0 \leq r \leq m, 0 \leq \theta \leq \pi/2\}.
\]

Although we have not proved it for two-dimensional integrals, if it is true that if $F, \ G \in BR(D)$ and

\[
0 \leq F(x, y) \leq G(x, y), \quad \text{for all } (x, y) \in D,
\]

then

\[
0 \leq \int \int_D F \leq \int \int_D G,
\]

where $D$ is any planar region. If, in particular, $S \subset D$ is a subset of $D$, if $G$ is positive and Riemann integrable over $D$, taking

\[
F(x, y) = \begin{cases} 
G(x, y) & \text{if } (x, y) \in S, \\
0 & \text{if } (x, y) \in D \setminus S,
\end{cases}
\]

then

\[
\int \int_S F = \int \int_D G,
\]

where $S$ is the new region of integration.
leads us to the conclusion that

\[ 0 \leq \int_{\mathcal{S}} G \leq \int_{\mathcal{D}} G, \quad \mathcal{S} \subset \mathcal{D}. \]

Now we apply this, taking

\[ G(x, y) = e^{-x^2 - y^2} \]

and \( \mathcal{S}, \mathcal{D} \) to be the regions of interest:

\[
\int_{\mathcal{D}(n)} e^{-x^2 - y^2} \leq \int_{\mathcal{S}(n)} e^{-x^2 - y^2} \leq \int_{\mathcal{D}(\sqrt{2}n)} e^{-x^2 - y^2}.
\]

Now we calculate

\[
\int_{\mathcal{D}(m)} e^{-x^2 - y^2} = \int_0^{\pi/2} d\theta \int_0^m e^{-r^2} r \, dr
\]

\[
= \left( \frac{\pi}{2} \right) \int_0^m e^{-r^2} r \, dr
\]

\[
= \left( \frac{\pi}{2} \right) \int_0^{m^2} \frac{1}{2} e^{-t} dt \quad \text{(substituting } r^2 = t \text{)}
\]

\[
= \left( \frac{\pi}{4} \right) (1 - e^{-m^2}),
\]

for any \( m > 0 \). Hence

\[
\left( \frac{\pi}{4} \right) (1 - e^{-n^2}) \leq \int_{\mathcal{S}(n)} e^{-x^2 - y^2} \leq \left( \frac{\pi}{4} \right) (1 - e^{-2n^2}).
\]

Since everything is positive we may take square roots freely, yielding

\[
\left( \sqrt{\pi}/2 \right) (1 - e^{-n^2})^{1/2} \leq \int_0^n e^{-x^2} \, dx \leq \left( \sqrt{\pi}/2 \right) (1 - e^{-2n^2})^{1/2},
\]

for all \( n > 0 \). Here we have substituted from our original equality.

Looking at this pair of inequalities, you can see that we have squeezed the desired integral between two explicit bounds. If we let \( n \to \infty \) and if the upper and lower bounds both converge to the same limit (they do), then the integral must also converge to that limit. Then

\[
\lim_{n \to \infty} \left( \sqrt{\pi}/2 \right) (1 - e^{-n^2})^{1/2} = \sqrt{\pi}/2 \quad \text{and} \quad \lim_{n \to \infty} \left( \sqrt{\pi}/2 \right) (1 - e^{-2n^2})^{1/2} = \sqrt{\pi}/2
\]

imply that

\[
\lim_{n \to \infty} \int_0^n e^{-x^2} \, dx = \int_0^\infty e^{-x^2} \, dx = \sqrt{\pi}/2.
\]

Now let us be completely oblivious of all mathematical rigour for a moment. We treat \( \infty \) as a number (?!?) and simply forge ahead. This is the type of calculation found in many texts on 'Mathematical Methods for Scientists and Engineers'.

To evaluate

\[
\int_0^\infty e^{-x^2} \, dx
\]

we observe that

\[
\left( \int_0^\infty e^{-x^2} \, dx \right)^2 = \int_0^\infty e^{-x^2} \, dx \int_0^\infty e^{-y^2} \, dy
\]

\[
= \int_{\mathcal{R}} e^{-x^2 - y^2},
\]

where \( \mathcal{R} \) is the first quadrant \( \{(x, y): x \geq 0, y \geq 0\} \).
Transforming to polar co-ordinates, the above integral becomes
\[
\int_{r=0}^{r=\infty} \int_{\theta=0}^{\theta=\pi/2} e^{-r^2} r \, dr \, d\theta = \left( \int_0^{\pi/2} \, d\theta \right) \times \left( \int_0^{\infty} r e^{-r^2} \, dr \right)
\]
\[
= \frac{\pi}{2} \left[ \frac{-e^{-r^2}}{2} \right]_0^{\infty} = \frac{\pi}{4}.
\]

As the integral is positive, having a positive integrand, we must take the positive square root. Hence we have the wonderful result
\[
\int_0^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.
\]

We now wish to evaluate the second integral. We need the standard result
\[
\int_0^m e^{-xy} \, dy = \frac{1}{x} (1 - e^{-mx}), \quad m, x > 0.
\]

Now if the Riemann integral allowed limits to be taken inside the integral, we could write
\[
\int_0^n \sin \frac{x}{x} \, dx = \lim_{m \to \infty} \int_0^n \sin \frac{x}{x} (1 - e^{-mx}) \, dx,
\]
and use Laplace transform methods; but we have no theorems allowing this. Such theorems hold for the Lebesgue integral, not the Riemann integral. However, we can follow this path using inequalities, leaving limits until after the integrals are evaluated. This brings us to consider the integral
\[
I_{n,m} = \int_0^n \sin \frac{x}{x} (1 - e^{-mx}) \, dx, \quad n, m > 0.
\]

Applying Fubini's theorem to the rectangle $[0, n] \times [0, m]$ we see that
\[
I_{n,m} = \int_0^n \int_0^m \sin \frac{x}{x} e^{-xy} \, dx \, dy = \int_0^m \int_0^n \sin \frac{x}{x} e^{-xy} \, dx \, dy.
\]

The integral over $x$ can be done: omitting details, we obtain
\[
I_{n,m} = \int_0^m \frac{1 - e^{-ny} \cos n - ye^{-ny} \sin n}{y^2 + 1} \, dy
\]
\[
= \tan^{-1} m - \cos n \int_0^m \frac{e^{-ny}}{y^2 + 1} \, dy - \sin n \int_0^m \frac{ye^{-ny}}{y^2 + 1} \, dy.
\]

It is now necessary to rearrange terms and apply various bounds. Let us start with the bounds.

If we note that
\[
|\sin n| \leq 1, \quad |\cos n| \leq 1
\]
and
\[
\sup_{[0,m]} \left( \frac{y + 1}{y^2 + 1} \right) < 2,
\]
then
\[
|\sin n \int_0^m \frac{ye^{-ny}}{y^2 + 1} \, dy + \cos n \int_0^m \frac{e^{-ny}}{y^2 + 1} \, dy| \leq \int_0^m \frac{y + 1}{y^2 + 1} \, e^{-ny} \, dy
\]
\[
\leq \sup_{[0,m]} \left( \frac{y + 1}{y^2 + 1} \right) \int_0^m \, e^{-ny} \, dy
\]
\[
< \left( \frac{2}{n} \right) (1 - e^{-mn}).
\]

The next bound we need uses the fact that
\[
\sup_{[0,n]} \left( \frac{\sin \frac{x}{x}}{x} \right) = 1.
\]
Then
\[ \int_0^\infty \frac{\sin x}{x} e^{-mx} \, dx \leq \int_0^\infty e^{-mx} \, dx = \frac{1}{m}(1 - e^{-mn}). \]

Now we go back to \( I_{m,n} \). We isolate the integral we want on one side of the equality, and we subtract \( \pi/2 \) from both sides. This is because we know that the integral will turn out to equal \( \pi/2 \).

Then
\[ \int_0^{\infty} \frac{\sin x}{x} \, dx - \frac{\pi}{2} = \int_0^{\infty} \frac{\sin x}{x} e^{-mx} \, dx - \frac{\pi}{2} + \tan^{-1} m - \cos n \int_0^{\infty} \frac{e^{-ny}}{y^2 + 1} \, dy - \sin n \int_0^{\infty} \frac{y e^{-ny}}{y^2 + 1} \, dy \]
\[ \leq \left| \int_0^{\infty} \frac{\sin x}{x} e^{-mx} \, dx \right| + \left| \frac{\pi}{2} - \tan^{-1} m \right| + \left| \cos n \int_0^{\infty} \frac{e^{-ny}}{y^2 + 1} \, dy \right| + \left| \sin n \int_0^{\infty} \frac{y e^{-ny}}{y^2 + 1} \, dy \right| \]
\[ \leq \frac{1}{m}(1 - e^{-mn}) + \left( \frac{\pi}{2} - \tan^{-1} m \right) + \frac{2}{n}(1 - e^{-mn}) \]
\[ < \frac{1}{m} + \frac{2}{n} + \frac{\pi}{2} - \tan^{-1} m, \]

for all \( n, m > 0 \). Notice that all dependence on \( m \) is on the right. This allows us to take the limit \( m \to \infty \), yielding
\[ \int_0^{\infty} \frac{\sin x}{x} \, dx - \frac{\pi}{2} < \frac{2}{n} \]

for all \( n > 0 \). Now taking the limit \( n \to \infty \) gives us the result
\[ \int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}. \]

Compare this calculation with the informal, non-rigorous approach. Now we treat \( \infty \) as a number and make use of Laplace transforms and Fubini's theorem.

It is easy to verify that
\[ \int_0^{\infty} e^{-xy} \, dy = \lim_{n \to \infty} \frac{1}{x}(1 - e^{-nx}) = \frac{1}{x} \quad (x > 0). \]

We substitute this into the integral we wish to evaluate, giving
\[ \int_0^{\infty} \frac{\sin x}{x} \, dx = \int_0^{\infty} \left( \int_0^{\infty} e^{-xy} \sin x \, dy \right) \, dx = \int_0^{\infty} \int_0^{\infty} e^{-xy} \sin x \, dx \, dy. \]

Now the integral
\[ \int_0^{\infty} e^{-xy} \sin x \, dx \]

is the Laplace transform of \( \sin x \) and can be evaluated by integrating by parts twice, to obtain
\[ \int_0^{\infty} e^{-xy} \sin x \, dx = \frac{1}{1 + y^2} \quad (y > 0). \]

It follows that
\[ \int_0^{\infty} \frac{\sin x}{x} \, dx = \int_0^{\infty} \frac{dy}{1 + y^2} = [\tan^{-1} y]_0^{\infty} = \frac{\pi}{2}. \]

We must insert a caveat here. It is not true that, if you apply an informal method and get a number out at the end, that you have found the value of the singular integral. Only under severe restrictions on the integrand is this true. This is why we stress the rigorous method. Having said that, in the next exercise and in the SAQs on this section, we ask you to use the informal method. In these cases the correct answer results.
5 Summary of the text

In this unit we have developed a definition of the Riemann integral based on the notions of least upper bound and greatest lower bound. We have established the basic properties of the integral, notably linearity and positivity. Following this pattern we have also constructed the definition of the double Riemann integral for regions of the plane.

We have shown that all bounded functions on \([a, b]\) which are continuous except at a finite set of points are Riemann integrable; we saw that this result still holds for functions that are continuous except for a finite set of points on bounded regions in the plane.

We have discussed the two Fundamental Theorems of Calculus, namely the Second:

\[
\int_a^b f' = f(b) - f(a), \quad f' \in BR[a, b],
\]

and the First, where the formula

\[
\int_a^b f = F(b) - F(a)
\]

defines a function \(F : [a, b] \rightarrow \mathbb{R}\) such that \(F' = f\) for \(f \in C[a, b]\).

For purposes of evaluation of integrals we have considered integration by parts and integration by substitution, in both one and two dimensions. For two dimensions we have presented and used the technique of repeated integrals and its justification through Fubini’s Theorem for Riemann integrals.

We have also introduced the basic inequality for integrals: if an integrable function is positive everywhere, its integral is positive. Amongst other things this led to Schwarz’s Inequality.

---

**Exercise 14**

Verify that

\[
\int_0^\infty e^{-xt}\sin x\,dx = \frac{1}{1+t^2} \quad (t > 0),
\]

using the informal non-rigorous method.

**Solution** Let

\[
F(t) = \int_0^\infty e^{-xt}\sin x\,dx.
\]

With \(u = e^{-xt}\) and \(dv = \sin x\,dx\), integration by parts gives

\[
F(t) = \left[ -xe^{-xt} \right]_0^\infty - \int_0^\infty te^{-xt}\cos x\,dx
\]

\[
= 1 - t\int_0^\infty e^{-xt}\cos x\,dx \quad (t > 0).
\]

(The condition \(t > 0\) ensures that \(e^{-xt} \rightarrow 0\) as \(x \rightarrow \infty\).)

Now set \(u = e^{-xt}\), \(dv = \cos x\,dx\) and use integration by parts again:

\[
F(t) = 1 - t \left( [e^{-xt}\sin x]_0^\infty + \int_0^\infty te^{-xt}\sin x\,dx \right)
\]

\[
= 1 - t^2 F(t).
\]

The result now follows by regrouping terms and clearing fractions.
Finally, you should note our remarks concerning assessment and the purpose of the unit, which we gave in the Introduction.

In the following subsections references to the set book or this text are preceded by $W$ or $T$ respectively. For example, a reference to page 9 of the set book is indicated by $[W9]$ and to Subsection 3.2 of this text by $[T3.2]$.

### 5.1 Notation

Note: $I$ is a reference to the Introduction.

$$P \quad [T1]$$
$$\delta x_i \quad [T1]$$
$$||P|| \quad [T1]$$
$$m_i \quad [T1]$$
$$M_i \quad [T1]$$
$$L(f, P) \quad [T1]$$
$$U(f, P) \quad [T1]$$
$$\int_a^b f \quad [T1]$$
$$\int_a^b f \quad [T1]$$
$$BR[a, b] \quad [T1]$$
$$\Delta(f, P) \quad [T1]$$
$$C[a, b] \quad [T1]$$
$$\alpha f \quad [T1]$$
$$f + g \quad [T1]$$
$$\int_a^\infty f \quad [T3]$$
$$\int_0^\infty e^{-st}f(t)dt \quad [T3]$$
$$BR[a, \infty) \quad [T3]$$
$$BR(-\infty, a] \quad BR(-\infty, +\infty) \quad [T3]$$
$$\mathcal{R} \quad [T4]$$
$$\iint_\mathcal{R} F \quad [T4]$$
$$\iiint_\mathcal{R} F \quad [T4]$$
$$\int_\mathcal{R} F \quad [T4]$$
$$\int_\mathcal{R} F \quad [T4]$$
$$BR(\mathcal{R}) \quad [T4]$$
$$\mathcal{D} \quad [T4]$$
$$V(F, \mathcal{D}) \quad [T4]$$
$$\int_c^d \left( \int_a^b F(x, y)dy \right)dx \quad [T4]$$
$$\int_c^d \left( \int_a^b F(x, y)dy \right)dx \quad [T4]$$
$$\partial(x, y) \quad \partial(t, u) \quad [T4]$$

### 5.2 Glossary

absolute convergence for integrals \quad [T3]
Comparison Test \quad [T3]
converge in the sense of mesh \quad [T1]
converge in the sense of refinement \quad [T1]
decreasing function \quad [T1]
double Riemann integrals \quad [T4]
exponential order with parameter $\alpha$ \quad [T3]
First Fundamental Theorem of Calculus \quad [T2]
Fubini's Theorem \quad [T4]
increasing function \quad [T1]
integration by parts \quad [T2]
integration by substitution \quad [T2]
Jacobian \quad [T4]
Laplace Transform \quad [T3]
lower Riemann integral \quad [T1]
lower Riemann sum \quad [T1]
length of an interval \quad [T1]
mesh of a partition \quad [T1]
monotonic function \quad [T1]
polar coordinates \quad [T4]
refinement \quad [T1]
repeated integral \quad [T4]
Riemann integral \quad [T1]
Riemann's $\Delta$-Criterion \quad [T1]
Schwarz's Inequality \quad [T2]
Second Fundamental Theorem of Calculus \quad [T2]
singular integral of the first kind \quad [T3]
singular integral of the second kind \quad [T3]
standard partition \quad [T1]
triangle inequality for integrals \quad [T2]
upper Riemann sum \quad [T1]
upper Riemann integral \quad [T1]
5.3 Results

Definition T.2.1(1)

(a) A partition $P$ of an interval $[a, b]$ is a finite collection of points of $[a, b],
\quad P = \{x_0, x_1, \ldots, x_n\},$
where $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$

We shall write $I_i = [x_{i-1}, x_i]$ for the $i$th interval defined by the adjacent points $x_{i-1}$ and $x_i$ of the partition.

The length of the $i$th interval is denoted by $\delta x_i = x_i - x_{i-1}.$

The mesh $\|P\| = \max_{1 \leq i \leq n} \{\delta x_i\}.$

A standard partition is a partition with equally spaced points. Hence all the $\delta x_i$ are the same, and so $\|P\|$ is equal to this common length. If there are $(n + 1)$ such points, then $\delta x_i = (b - a)/n$ and $\|P\| = (b - a)/n.$

(b) Let $f$ be bounded on $[a, b]$, let $P$ be a partition of $[a, b],$ and let 
\[ m_i = \inf_{I_i} f \quad \text{and} \quad M_i = \sup_{I_i} f. \]

The lower Riemann sum of $f$ is
\[ L(f, P) = \sum_{i=1}^{n} m_i \delta x_i. \]

The upper Riemann sum of $f$ is
\[ U(f, P) = \sum_{i=1}^{n} M_i \delta x_i. \]

(c) The lower (Riemann) integral of $f$ is
\[ \int_a^b f = \sup_{P} \{L(f, P)\}. \]

The upper (Riemann) integral of $f$ is
\[ \int_a^b f = \inf_{P} \{U(f, P)\}. \]

As $f$ is bounded, the lower and upper integrals both exist. It can be shown — and it seems intuitively correct — that
\[ \int_a^b f \leq \int_a^b f. \]

It is not always the case that these integrals are equal. We say that $f$ is (Riemann) integrable on $[a, b]$ if
\[ \int_a^b f = \int_a^b f; \]
in this case, the (Riemann) integral of $f$ on $[a, b]$ is
\[ \int_a^b f = \int_a^b f = \int_a^b f. \]

The class of integrable functions defined in this way is called the (bounded) Riemann integrable functions. We denote this class of functions by $BR[a, b].$
Definition T.2.1(2)

(a) A partition $Q$ is a refinement of a partition $P$ if $P$ is a subset of $Q$, and we write $P \subset Q$. Observe that if $x_{i-1}$ and $x_i$ are adjacent points in $Q$, the interval they define, $[x_{i-1}, x_i]$, is contained in one of the intervals $[y_{j-1}, y_j]$ determined by $P$. It may be that $x_{i-1}$ or $x_i$ is an end-point of $[y_{j-1}, y_j]$, but it may be that $[x_{i-1}, x_i]$ is entirely interior to $[y_{j-1}, y_j]$. If you draw a simple diagram you will quickly see this.

(b) The upper Riemann sums $U(f, P)$ for a bounded function $f$ on $[a, b]$ converge to the number $U(f)$ in the sense of refinement if, given any $\varepsilon > 0$, there is a partition $P(\varepsilon)$ such that

$$|U(f, P) - U(f)| < \varepsilon$$

for all partitions $P$ which are refinements of $P(\varepsilon)$.

Similarly, the lower Riemann sums $L(f, Q)$ converge in the sense of refinement to the limit $L(f)$ if, given any $\varepsilon > 0$, there exists a partition $Q(\varepsilon)$ such that

$$|L(f) - L(f, Q)| < \varepsilon$$

for all partitions $Q$ which are refinements of $Q(\varepsilon)$.

(c) We define a second type of limit. Keeping the previous notation, the $U(P, f)$ converge to a limit $U'(f)$ in the sense of mesh if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|U(f, P) - U'(f)| < \varepsilon$$

for all partitions $P$ with mesh $||P|| < \delta$. In other words, we wish $U(f, P)$ to converge to $U'(f)$ as $||P||$ tends to zero. A similar definition holds for the mesh convergence of the lower Riemann sums $L(f, Q)$ to a limit $L'(f)$.

Theorem T.2.1 If a function $f$ is Riemann integrable in the sense of Definition T.2.1(1), then the four numbers $U(f), L(f), U'(f), L'(f)$ all exist and equal the Riemann integral of $f$.

In the opposite direction, if either pair $U(f), L(f)$ or $U'(f), L'(f)$ exists and the two members of the pair are equal, then $f$ is Riemann integrable.

It follows that if $f \in BR[a, b]$, then

$$\lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n) = \int_a^b f,$$

where $P_n$ is the standard partition of mesh $||P_n|| = (b - a)/n$.

Theorem T.2.1.1 (Riemann's Δ-Criterion) Let $f$ be a bounded function on $[a, b]$. A necessary and sufficient condition for $f$ to be in $BR[a, b]$ is that, given any $\varepsilon > 0$, there is a partition $P$ of $[a, b]$ such that

$$\Delta(f, P) = U(f, P) - L(f, P) < \varepsilon,$$

where $U(f, P), L(f, P)$ are the upper and lower Riemann sums, respectively, and $P$ will depend on $\varepsilon$ in general.

Theorem T.2.1.2 Any continuous function is Riemann integrable; that is, $C[a, b] \subset BR[a, b]$, where we write $C[a, b]$ for the set of all continuous functions

$$f : [a, b] \to \mathbb{R}.$$  

Proposition T.2.1.3 Let $f$ be bounded on $[a, b]$ and continuous at all but a finite number of points of $[a, b]$. Then $f$ is Riemann integrable over $[a, b]$.
Proposition T.2.1.4

(a) If $f, g \in BR[a, b]$, then $f + g \in BR[a, b]$, where we define $f + g$ as the function whose rule is

\[ (f + g)(x) = f(x) + g(x). \]

(b) If $f \in BR[a, b]$ and $\alpha \in \mathbb{R}$, then $\alpha f \in BR[a, b]$, where we define $\alpha f$ to be the function whose rule is

\[ (\alpha f)(x) = \alpha f(x). \]

(c) If $f, g \in BR[a, b]$, then $fg \in BR[a, b]$, where $fg$ is the function whose rule is

\[ (fg)(x) = f(x)g(x). \]

If $|g(x)| \geq \delta > 0$ for some $\delta$ and all $x \in [a, b]$, and if we define $f/g$ to be the function whose rule is

\[ (f/g)(x) = f(x)/g(x), \]

then

\[ f/g \in BR[a, b]. \]

(d) If $f \in BR[a, b]$ and $[c, d] \subset [a, b]$, and if we define the restriction of $f$ to $[c, d]$ to be the function $g : [c, d] \rightarrow \mathbb{R}$ given by

\[ g(x) = f(x) \quad \text{if } x \in [c, d], \]

then $g \in BR[c, d]$. The function $g$ is often written as $f|_{[c,d]}$.

(e) If $f \in BR[a, b]$ and $g \in BR[b, c]$, then $h \in BR[a, c]$, where $h : [a, c] \rightarrow \mathbb{R}$ is given by

\[ h(x) = \left\{ \begin{array}{ll}
  f(x) & \text{if } a \leq x < b, \\
  g(x) & \text{if } b < x \leq c.
\end{array} \right. \]

The value of $h$ at $x = b$ is unimportant; other choices also yield functions which are Riemann integrable.

(f) If $f \in BR[a, b]$, then $|f| \in BR[a, b]$, where $|f| : [a, b] \rightarrow \mathbb{R}$ is given by

\[ |f|(x) = |f(x)| \quad \text{for all } x \in [a, b]. \]

Proposition T.2.2.1 Using the definitions given in Proposition T.2.1.4, the following Combination Rules hold for Riemann integrals.

(a) If $f, g \in BR[a, b]$, then

\[ \int_{a}^{b} (f + g) = \int_{a}^{b} f + \int_{a}^{b} g. \]

(b) If $f \in BR[a, b]$ and $\alpha \in \mathbb{R}$, then

\[ \int_{a}^{b} (\alpha f) = \alpha \int_{a}^{b} f. \]

(c) If $c$ is an intermediate point of $[a, b]$, that is, $a \leq c \leq b$, and $f \in BR[a, b]$, then

\[ \int_{a}^{b} f = \int_{a}^{c} f + \int_{c}^{b} f. \]

We interpret $\int_{c}^{c} f = 0$, as a convention.

(d) It is consistent with the other definitions and results to define

\[ \int_{b}^{a} f = -\int_{a}^{b} f \]

for $a \leq b$ and $f \in BR[a, b]$. 

Page 56
(e) If \( x, y, z \) are any three points in any order in \([a, b]\), then for any \( f \in BR[a, b] \),
\[
\int_x^y f + \int_y^z f = \int_x^z f.
\]

**Proposition T.2.2.2** If \( f \in BR[a, b] \) and if \( f(x) \geq 0 \) for all \( x \in [a, b] \), then
\[
\int_a^b f \geq 0.
\]

**Corollary T.2.2.2** If \( f, g \in BR[a, b] \) and if \( f(x) \leq g(x) \) for all \( x \in [a, b] \) then
\[
\int_a^b f \leq \int_a^b g.
\]

**Theorem T.2.2.2(1) (The Triangle Inequality for Integrals)** For any \( f \in BR[a, b] \),
\[
\left| \int_a^b f \right| \leq \int_a^b |f|.
\]

**Theorem T.2.2.2(2)** For any two functions \( f, g \) in \( BR[a, b] \),
\[
\left( \int_a^b fg \right)^2 \leq \int_a^b f^2 \int_a^b g^2.
\]

**Theorem T.2.2.3 (The Second Fundamental Theorem of Calculus)** If \( f : [a, b] \rightarrow \mathbb{R} \) has a derivative \( f' \) which is Riemann integrable, that is, if \( f' \in BR[a, b] \), then
\[
\int_a^b f' = f(b) - f(a).
\]

**Proposition T.2.2.3**
(a) **(Integration by Parts)** If \( f, g : [a, b] \rightarrow \mathbb{R} \) have derivatives \( f', g' \in BR[a, b] \), respectively, then
\[
\int_a^b f'g = [f(x)g(x)]_a^b - \int_a^b fg'.
\]

(b) **(Integration by Substitution)** If \( f : [a, b] \rightarrow \mathbb{R} \) has a derivative, \( f' \in BR[a, b] \); if \( g : [\alpha, \beta] \rightarrow \mathbb{R} \) is one-one, has a derivative \( g' \in BR[\alpha, \beta] \), if \( g(\alpha) = a \), \( g(\beta) = b \); and if \((f' \circ g)g' \in BR[\alpha, \beta] \), then
\[
\int_a^b f' = \int_\alpha^\beta (f' \circ g)g'.
\]

In Leibniz notation this reads
\[
\int_a^b f(x)dx = \int_\alpha^\beta f'(t)[g'(t)]dt.
\]

**Theorem T.2.2.4 (The First Fundamental Theorem of Calculus)** If \( f \in C[a, b] \) and if we define the function \( F : [a, b] \rightarrow \mathbb{R} \) by the rule
\[
F(x) = \int_a^x f,
\]
then \( F \) is differentiable and \( F' = f \).
Definition T.2.3.1 We say that the integral
\[ \int_a^\infty f \]
converges if for each \( b > a \) the non-singular integral
\[ \int_a^b f \]
exists and the limit
\[ \lim_{b \to \infty} \int_a^b f \]
extists. The value of the singular integral is then defined to be
\[ \int_a^\infty f = \lim_{b \to \infty} \int_a^b f. \]
In a similar way we define
\[ \int_{-\infty}^b f = \lim_{a \to -\infty} \int_a^b f, \]
when it exists. However, and this is surprising, we can only define
\[ \int_{-\infty}^{+\infty} f = \lim_{b \to \infty} \int_{-b}^b f + \lim_{a \to -\infty} \int_a^0 f = \int_0^\infty f + \int_{-\infty}^0 f, \]
if both \( \int_0^\infty f \) and \( \int_{-\infty}^0 f \) exist. The reason is that \( \lim_{c \to \infty} \int_{-c}^c f \) can exist without both the integrals \( \int_0^\infty f \) and \( \int_{-\infty}^0 f \) existing. Any of the three integrals, \( \int_0^\infty f \), \( \int_{-\infty}^0 f \), or \( \int_{-\infty}^{+\infty} f \), is said to be a singular integral of the first kind, if it exists.

The notations \( BR[a, \infty) \), \( BR(-\infty, b] \) and \( BR(-\infty, +\infty) \) are used for the sets of functions integrable over \([a, \infty)\), \((-\infty, b]\) and \((-\infty, +\infty)\), respectively.

Proposition T.2.3.1 (The Comparison Test) If \( 0 \leq f(x) \leq g(x) \) for all \( x \geq a \), if \( f \in BR[a, b] \) for all \( b \geq a \), and if we know that
\[ \int_a^\infty g \]
converges, then it follows that
\[ \int_a^\infty f \]
must converge too.

Definition T.2.3.2 Given that
\[ \int_{a+\epsilon}^b f \]
exists as a non-singular integral for every \( \epsilon > 0 \), where \( f \) is not necessarily bounded on \([a, b]\), we say that
\[ \int_a^b f \]
converges if the limit
\[ \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f \]
extists. If it does, its value is given by the limit
\[ \int_a^b f = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f, \]
and is said to be a singular integral of the second kind.
If \( f \) is bounded, then \( \int_a^b f \) exists as a singular integral of the second kind if and only if \( f \in BR[a, b] \), in which case the two notions of integral coincide. In this sense, the definition of a singular integral is a sensible extension of the Riemann integral.

**Proposition T.2.3.2**  If \( 0 \leq f(x) \leq g(x) \) for all \( x \in [a, b] \), if \( f \in BR[a+\varepsilon, b] \) for all \( \varepsilon > 0 \), and if we know that

\[
\int_a^b g = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b g
\]

converges as a singular integral of the second kind, then

\[
\int_a^b f = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f
\]

converges as a singular integral of the second kind, with

\[
\int_a^b f \leq \int_a^b g.
\]

**Definition T.2.3.3**  For any function \( f : [a, \infty) \to \mathbb{R} \) such that \( f \in BR[a, b] \) for all \( b \geq a \), we say that the singular integral

\[
\int_a^\infty f
\]

is *absolutely convergent* whenever the integral

\[
\int_a^\infty |f|
\]

converges.

**Proposition T.2.3.3**  (*The Absolute Convergence Test*)  Let \( f \in BR[a, b] \) for all \( b \geq a \). If the singular integral

\[
\int_a^\infty f
\]

is absolutely convergent, then it is convergent.

**Definition T.2.3.4**

(a)  For a function \( f : [0, \infty) \to \mathbb{R} \), its *Laplace transform* is defined to be the function

\[
F(x) = \int_0^\infty e^{-xt} f(t) dt
\]

over the domain of \( x \) in \( \mathbb{R} \) for which the integral exists. The integral is to be interpreted as a singular Riemann integral of the first kind. If no such domain of \( x \) exists, we say that \( f \) has no Laplace transform.

(b)  We say that a function \( f : [0, \infty) \to \mathbb{R} \) has *exponential order* with parameter \( \alpha \) if there is a positive constant \( M \) such that

\[
|f(t)| \leq Me^{\alpha t}
\]

for all \( t \geq 0 \). When \( \alpha = 1 \) we often abbreviate this to ‘\( f \) is of exponential order’.

**Theorem T.2.3.5**  (*Existence of Laplace Transforms*)  If \( f \) has exponential order with parameter \( \alpha \) and \( f \in BR[0, T] \) for all \( T \geq 0 \), then its Laplace transform converges absolutely for all \( x > \alpha \). Hence \( F(x) \) exists for all \( x > \alpha \).
Definition T.2.4.1

(a) A partition of $R = [a, b] \times [c, d]$ is an ordered pair $(P', P'')$, where $P'$ is a partition of $[a, b]$ and $P''$ is a partition of $[c, d]$.

If we write $P' = \{x_0, x_1, \ldots, x_n\}$ and $P'' = \{y_0, y_1, \ldots, y_m\}$, then we have intervals $[I_i'] = [x_{i-1}, x_i]$ and $[I_j'] = [y_{j-1}, y_j]$ of $[a, b]$ and $[c, d]$, respectively, as we did in Definition T.2.1(1). Similarly, $(P', P'')$ defines the rectangles

$$R_{ij} = I_i' \times I_j' \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,$$

which are subrectangles of $R$.

(b) Let $F : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be bounded and let $Q$ be a partition of $[a, b] \times [c, d]$.

For each $i, j$ we define the least upper bound and the greatest lower bound of $F$ on the subrectangle $R_{ij}$ by the rules

$$M_{ij} = \sup\{F(x, y) : (x, y) \in R_{ij}\}$$
$$m_{ij} = \inf\{F(x, y) : (x, y) \in R_{ij}\}.$$

Summing over all the subrectangles we define the upper and lower Riemann sums, respectively:

$$U(F, Q) = \sum_{j=0}^{n} \sum_{i=0}^{m} M_{ij} (x_i - x_{i-1})(y_j - y_{j-1}),$$
$$L(F, Q) = \sum_{j=0}^{n} \sum_{i=0}^{m} m_{ij} (x_i - x_{i-1})(y_j - y_{j-1}).$$

(c) The lower Riemann integral of $F$ is

$$\int_{R} F = \sup_{Q} \{L(F, Q)\}$$

and the upper Riemann integral of $F$ is

$$\int_{R} F = \inf_{Q} \{U(F, Q)\}.$$

These upper and lower bounds are over all partitions $Q$ of the closed rectangle $R$.

As $F$ is bounded, the lower and upper Riemann integrals both exist, and (as can be shown)

$$\int_{R} F \leq \int_{R} F.$$

We say that $F$ is Riemann integrable over $R$ if both these integrals are equal, and in this case we say that their common value is the Riemann integral of $F$ over $R$, and write

$$\int_{R} F = \int_{R} F = \int_{R} F.$$

The set of all Riemann integrable functions on $R$ is written as $BR(R)$.

Remarks T.2.4.1

(a) If $D$ is a bounded non-rectangular region it can be enclosed in a closed rectangle $R$. It does not matter which rectangle $R$ we consider, only that $D \subset R$.

We say that $F$ is Riemann integrable on $D$, $F \in BR(D)$ if and only if $G \in BR(R)$, and define

$$\int_{D} F = \int_{R} G.$$

It is elementary that this definition is independent of the choice of enclosing rectangle $R$. 
It is interesting to note that if we take $F(x, y) = 1$ for $(x, y) \in \mathcal{D}$ and zero otherwise. Then

$$\int \int_{\mathcal{D}} F = A(\mathcal{D})$$

is the area of $\mathcal{D}$, if it exists.

(b) For unbounded sets in the plane, $\mathcal{D} \subset \mathbb{R}^2$, we proceed as for singular integrals of the first kind. Namely, we consider an increasing sequence $\mathcal{D}_n$ of bounded regions, $\mathcal{D}_n \subset \mathcal{T} > \mathcal{D}_{n+1}$, such that as $n$ increases indefinitely, these regions eventually approach $\mathcal{D}$. We then compute the integrals $\int \int_{\mathcal{D}_n} F$, and if these converge as $n \to \infty$, the limit is defined to be the integral $\int \int_{\mathcal{D}} F$.

We can also define a notion of volume based on the Riemann integral. If $F : \mathcal{D} \to \mathbb{R}$ is Riemann integrable over the planar region $\mathcal{D} \subset \mathbb{R}^2$, so $F \in BR(\mathcal{D})$, we interpret

$$\int \int_{\mathcal{D}} F = V(F, \mathcal{D})$$

as the volume of the three-dimensional region interior to the figure bounded by $\mathcal{D}$ (as the 'base') and the values of $F$ (as the 'height').

**Theorem T.2.4.2** *(The Repeated Integral Theorem)*  If $F$ is integrable on $\mathcal{R} = [a, b] \times [c, d]$ and $\int_a^b F(x, y)dx$ exists for each value of $y \in [c, d]$, then the repeated integral

$$\int_c^d \int_a^b F(x, y)dx dy$$

exists, and equals the Riemann integral of $F$ over $\mathcal{R}$:

$$\int \int_{\mathcal{R}} F = \int_c^d \int_a^b F(x, y)dx dy.$$

**Corollary T.2.4.2(1) (Fubini's Theorem)**  If in addition to the conditions of Theorem T.2.4.2 we assume that $\int_a^b F(x, y)dy$ exists for each $x \in [a, b]$ then we have

$$\int \int_{\mathcal{R}} F = \int_a^b \int_c^d F(x, y)dx dy = \int_a^b \int_c^d F(x, y)dy dx.$$

**Corollary T.2.4.2(2)** If $F$ is continuous on $\mathcal{R}$, then $\lim \int \int_{\mathcal{R}} F, \int_a^b F(x, y)dy$ for all $x \in [a, b]$, and $\int_a^c F(x, y)dx$ for all $y \in [c, d]$ all exist and so are equal.

**Proposition T.2.4.3** *(Two-Dimensional Substitution Rule)*  A change of variables in a two-dimensional integral

$$\int \int_{\mathcal{D}} f(x, y)dx dy$$

involves replacing $x, y$ by two functions $x(t, u), y(t, u)$ of two variables $t, u$ subject to appropriate conditions. The analogue of $dz/dt$ in one variable is the so-called Jacobian determinant

$$\frac{\partial (x, y)}{\partial (t, u)} = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{vmatrix}.$$

The formula is

$$\int \int_{\mathcal{D}} f(x, y)dx dy = \int \int_{\mathcal{S}} f(x(t, u), y(t, u)) \left| \frac{\partial (x, y)}{\partial (t, u)} \right| dt du,$$

where $\mathcal{S}$ is the region of integration expressed in terms of the new variables $t, u$. 

51
6 Self-Assessment Questions

6.1 Integrating inequalities

SAQ 1
Integrate the inequality
\[ 1 - \frac{x}{\pi} \leq \frac{\sin x}{x} \leq 1 \]
over the interval \( 0 < x < \pi \) to obtain
\[ \frac{\pi}{2} \leq \int_0^\pi \frac{\sin x}{x} \, dx \leq \pi. \]

SAQ 2
Integrate the inequality
\[ 1 - x^2 \leq e^{-x^2} \leq 1 - x^2 + \frac{1}{2} x^4 \]
over the interval \( 0 < x < 1 \) to obtain
\[ \frac{2}{3} \leq \int_0^1 e^{-x^2} \, dx \leq \frac{23}{30}. \]

6.2 Schwarz's Inequality

SAQ 3
Verify that Schwarz's Inequality applied to the integral
\[ \int_0^{\pi/4} \sec x \tan x \, dx \]
gives the inequality
\[ (\sqrt{2} - 1)^2 \leq 1 - \frac{\pi}{4}. \]
Deduce that \( \pi \leq 8(\sqrt{2} - 1). \)
Calculate both sides of this inequality and compare their values.

SAQ 4
Verify that Schwarz's Inequality applied to the integral
\[ \int_1^2 \frac{\log x}{x} \, dx \]
gives
\[ \frac{(\log 2)^2}{2} \leq 1 - \log 2. \]
Deduce that \( \log 2 \leq \sqrt{3} - 1. \)
Calculate both sides of this inequality and compare their values.

6.3 Singular integrals

SAQ 5
Which of the following integrals converge? Find the values of those which converge.
For any which do not converge, supply a brief reason.
(a) \[ \int_{0}^{\infty} xe^{-x^2} \, dx \]
(b) \[ \int_{1}^{\infty} \frac{\log x}{x} \, dx \]
(c) \[ \int_{0}^{1} \frac{dx}{x \sqrt{x}} \]
(d) \[ \int_{0}^{1} \frac{\log x}{\sqrt{x}} \, dx \]

**SAQ 6**
Which of the following integrals converge absolutely? For any which do not converge absolutely, supply a brief reason.

(a) \[ \int_{0}^{\infty} \cos x \, dx \]
(b) \[ \int_{1}^{\infty} \frac{\cos x}{x^3} \, dx \]

### 6.4 Double integrals

**SAQ 7**

Show that
\[ \int \int_{\mathcal{R}} \sin(x + 2y) \, dx \, dy = 1, \]
where \( \mathcal{R} \) is the square region \( \{(x, y): 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2\} \).

**SAQ 8**

Evaluate the integral
\[ \int \int_{\mathcal{R}} h \left( 1 - \frac{\sqrt{x^2 + y^2}}{a} \right) \, dx \, dy, \]
where \( \mathcal{R} \) is the circular region \( \{(x, y): x^2 + y^2 \leq a^2\} \) and \( h \) is a real number, by changing to polar coordinates.

Deduce that the volume of a cone with radius \( a \) and height \( h \) is \( \frac{1}{3} \pi a^2 h \).

### 6.5 Laplace transforms

**SAQ 9**

Verify, using the informal, non-rigorous method, that the functions \( \cos t \), \( \cos 2t \), \( \sin^2 t \) have the Laplace transforms indicated below:

(a) \[ \int_{0}^{\infty} e^{-xt} \cos t \, dt = \frac{x}{1 + x^2} \]
(b) \[ \int_{0}^{\infty} e^{-xt} \cos 2t \, dt = \frac{x}{4 + x^2} \]
(c) \[ \int_{0}^{\infty} e^{-xt} \sin^2 t \, dt = \frac{2}{x(1 + x^2)} \]
Show that
\[ \int_0^\infty \frac{\sin^2 x}{x^2} \, dx = \frac{\pi}{2} \]
by substituting (see Exercise 11)
\[ \frac{1}{x^2} = \int_0^\infty t \, e^{-xt} \, dt \]
and using Fubini's Theorem. Again, use the informal, non-rigorous method.

**Solutions to Self-Assessment Questions**

**Solution 1**
From the inequality and the answer we are supposed to verify, we see that we must integrate \( 1 - \left(\frac{x}{\pi}\right) \) and \( 1 - \left(\frac{x^2}{\pi^2}\right) \) over the interval \([0, \pi]\). This is quite straightforward, and we find
\[ \int_0^\pi \left(1 - \frac{x}{\pi}\right) \, dx = \left[ x - \frac{x^2}{2\pi} \right]_0^\pi = \frac{\pi}{2} \]
and
\[ \int_0^\pi \, dx = [x]_0^\pi = \pi. \]
Thus the inequality leads to the required answer.

**Solution 2**
The same strategy used in SAQ 1 works here. The two integrals we require are
\[ \int_0^1 (1 - x^2) \, dx = \left[ x - \frac{1}{3}x^3 \right]_0^1 = \frac{2}{3} \]
and
\[ \int_0^1 (1 - x^2 + \frac{1}{3}x^4) \, dx = \left[ x - \frac{1}{3}x^3 + \frac{1}{10}x^5 \right]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} = \frac{33}{30}, \]
respectively.

**Solution 3**
There are two parts to this question. The first requires us to evaluate the \( \sec x \tan x \) integral. This gives us
\[ \int_0^{\pi/4} \sec x \tan x \, dx = [\sec x]_{\pi/4}^0 = \sqrt{2} - 1. \]
The second part is to evaluate the integrals necessary for Schwarz's Inequality. We pick, obviously, \( f(x) = \sec x \) and \( g(x) = \tan x \). The integrals of their squares are
\[ \int_0^{\pi/4} \sec^2 x \, dx = [\tan x]_{\pi/4}^0 = 1, \]
\[ \int_0^{\pi/4} \tan^2 x \, dx = \int_0^{\pi/4} (\sec^2 x - 1) \, dx = [\tan x - x]_{\pi/4}^0 = 1 - \frac{\pi}{4}, \]
respectively.
Schwarz's Inequality therefore gives 
\[(\sqrt{2} - 1)^2 \leq 1 - \frac{\pi}{4}.\]
The next inequality in the question arises from rearrangement of this inequality. Thus,
\[2 - 2\sqrt{2} + 1 \leq 1 - \frac{\pi}{4},\]
\[\frac{\pi}{4} \leq 2\sqrt{2} - 2 = 2(\sqrt{2} - 1),\]
\[\pi \leq 8(\sqrt{2} - 1).\]
To two significant figures, the calculator gives \(\pi = 3.14\ldots, 8(\sqrt{2} - 1) = 3.31\ldots\)

**Solution 4**
The same strategy applies here as in the previous SAQ. Setting \(f(x) = 1/x\) and \(g(x) = \log x\) we start by doing the integral of \(f \times g\). This requires a change of variables substitution, as indicated:
\[
\int_1^2 \frac{\log x}{x} \, dx = \int_0^{\log 2} u \, du \quad (u = \log x)
\]
\[= \left[ \frac{1}{2} u^2 \right]_0^{\log 2} = \frac{(\log 2)^2}{2}.
\]
We next evaluate the integrals of \(f^2\) and \(g^2\). We find
\[
\int_1^2 (\log x)^2 \, dx = [x(\log x)^2]_1^2 - 2 \int_1^2 \log x \, dx
\]
\[= 2(\log 2)^2 - 2[x \log x]_1^2 + 2 \int_1^2 \, dx
\]
\[= 2(\log 2)^2 - 4\log 2 + 2
\]
\[= 2(\log 2 - 1)^2
\]
and
\[
\int_1^2 \frac{dx}{x^2} = \left[ -\frac{1}{x} \right]_1^2 = \frac{1}{2}.
\]
Schwarz's Inequality therefore gives 
\[
\left( \frac{(\log 2)^2}{2} \right)^2 \leq (1 - \log 2)^2
\]
and so
\[
\frac{(\log 2)^2}{2} \leq 1 - \log 2.
\]
It follows by rearrangement that
\[
(\log 2)^2 \leq 2 - 2\log 2,
\]
\[
(\log 2)^2 + 2\log 2 + 1 \leq 3,
\]
\[
(\log 2 + 1)^2 \leq 3,
\]
\[
\log 2 + 1 \leq \sqrt{3},
\]
\[
\log 2 \leq \sqrt{3} - 1.
\]
The calculator gives \(\log 2 = 0.69\ldots, \sqrt{3} - 1 = 0.73\ldots\) to two significant figures.

**Solution 5**
We are asked for a reason for non-convergence as well as values when convergence occurs. The only way full credit can be obtained is to 'do' the integrals. Simply stating, for example, '(c) does not converge because it is singular' is not acceptable.
In 'doing' the integrals we have indicated any substitutions we have made.

(a) By substituting $u = a^2$, we have

$$
\int_0^b x e^{-x^2} dx = \frac{1}{2} \int_0^{b^2} e^{-u} du \quad (u = x^2)
$$

$$
= \frac{1}{2} \left[-e^{-u}\right]_0^{b^2} = \frac{1}{2} \left(1 - e^{-b^2}\right) \to \frac{1}{2}
$$

as $b \to \infty$. Therefore the integral converges, with value $\frac{1}{2}$.

(b) By substituting $u = \log x$ we have

$$
\int_1^b \log x \cdot \frac{1}{x} dx = \int_0^b \log u \cdot \frac{1}{u} du \quad (u = \log x)
$$

$$
= \left[\frac{1}{2} u^2\right]_0^b = \frac{1}{2} (\log b)^2 \to \infty
$$

as $b \to \infty$. Therefore the integral diverges.

A brief reason for the divergence is that $\frac{\log x}{x}$ does not tend to zero fast enough as $b \to \infty$. Alternatively, a comparison of $\log x/x$ with $1/x$ would lead to the same conclusion.

(c) Substituting $y = \sqrt{x}$ yields

$$
\int_0^1 \frac{dx}{x^{1/2}} = \int_0^1 \frac{2}{y^{1/2}} dy = \left[-\frac{2}{y^{1/2}}\right]_0^1
$$

$$
= 2 \left(1 - \frac{1}{\sqrt{\varepsilon}}\right) \to \infty
$$

as $\varepsilon \to 0^+$. Hence the integral diverges.

A brief reason for the divergence is that $x^{1/2}$ approaches zero too rapidly as $x \to 0^+$.

(d) We integrate by parts, setting $u = \log x$ and $dv = dx/\sqrt{x}$. Then

$$
\int_\varepsilon^1 \log x \cdot \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x} \log x\right]_\varepsilon^1 - 2 \int_\varepsilon^1 \frac{dx}{\sqrt{x}}
$$

$$
= -2\sqrt{\varepsilon} \log \varepsilon - 2 \left[2\sqrt{x}\right]_\varepsilon^1
$$

$$
= -2\sqrt{\varepsilon} \log \varepsilon - 4(1 - \sqrt{\varepsilon}) \to -4
$$

as $\varepsilon \to 0$. Hence the integral converges, with value $-4$.

**Solution 6**

(a) We will compare $|\cos \varepsilon|$ with $\cos^2 \varepsilon$ and apply the Comparison Test of Subsection 3.1. Now

$$
|\cos \varepsilon| \geq \cos^2 \varepsilon.
$$

We can show that

$$
\int_0^\infty \cos^2 \varepsilon \, d\varepsilon
$$

diverges, since

$$
\int_0^b \cos^2 \varepsilon \, d\varepsilon = \frac{1}{2} \int_0^b (1 + \cos 2\varepsilon) \, d\varepsilon
$$

$$
= \frac{1}{2} \left[x + \frac{1}{2} \sin 2\varepsilon\right]_0^b
$$

$$
= \frac{1}{2} b + \frac{1}{4} \sin 2b \to \infty
$$

as $b \to \infty$. Therefore, by the Comparison Test the given integral is not absolutely convergent.

You might wish to know why we did not integrate $|\cos \varepsilon|$ directly. This is possible, but needs extra thought. For large $b$, there is an integer $N$ so that $b = N\pi/2 + c$, where $0 \leq c < \pi/2$. Integrating over intervals of length $\pi/2$ makes things easiest.
Let us write
\[ \int_{0}^{b} |\cos x| dx = \int_{0}^{\pi/2} |\cos x| dx + \int_{\pi/2}^{\pi} |\cos x| dx + \cdots + \int_{N\pi/2}^{N\pi/2+c} |\cos x| dx. \]

Now for all \(0 \leq m \leq N-1\) we have
\[ \int_{m\pi/2}^{(m+1)\pi/2} |\cos x| dx = \int_{0}^{\pi/2} |\cos x| dx = [\sin x]_{0}^{\pi/2} = 1, \]
and
\[ \int_{N\pi/2}^{N\pi/2+c} |\cos x| dx \leq \int_{N\pi/2}^{(N+1)\pi/2} |\cos x| dx = 1. \]

We also note that \(b > N\). Putting everything together,
\[ N \leq b < \int_{0}^{b} |\cos x| dx = 1 + \cdots + 1 + \int_{N\pi/2}^{N\pi/2+c} |\cos x| dx, \]
where we have used that fact that \(|\cos x| \leq 1\). Then
\[ N < \int_{0}^{b} |\cos x| dx \leq N + 1. \]

From \(b = N\pi/2 + c\) it follows that \(b \to \infty\) as \(N \to \infty\), so taking the limit \(N \to \infty\) we see that the integral does not converge.

(b) We compare \(|\cos x|/x^3\) with \(1/x^3\), since
\[ \left| \frac{\cos x}{x^3} \right| \leq \frac{1}{x^3}. \]

Now the integral
\[ \int_{1}^{\infty} \frac{dx}{x^3} \]
converges, since
\[ \int_{1}^{b} \frac{dx}{x^3} = \left[ -\frac{1}{2x^2} \right]_{1}^{b} = \frac{1}{2} \left( 1 - \frac{1}{b^2} \right) \to \frac{1}{2} \]
as \(b \to \infty\). Therefore the given integral converges absolutely, by the Comparison Test.

**Solution 7**

The region \(R\) is a square, so the first of the repeated integrals has limits 0, \(\pi/2\). We do the integration over \(x\) first. We have indicated some cross-sections of \(\sin(x + 2y)\) for particular values of \(y\). In all cases one integrates \(\sin(x + a)\) over the interval \([0, \pi/2]\). The result, with \(a = 2y\), is a function \(A(y)\) of \(y\) (the area of each cross-sectional slice), which is then integrated over \([0, \pi/2]\).
Then
\[ A(y) = \int_0^{\pi/2} \sin(x + 2y) \, dx \]
\[ = \int_0^{\pi/2} [-\cos(x + 2y)] \, dy \]
\[ = \cos 2y + \sin 2y. \]

It now follows that
\[
\int \int_{R} \sin(x + 2y) \, dx \, dy = \int_0^{\pi/2} A(y) \, dy
\]
\[ = \int_0^{\pi/2} (\cos 2y + \sin 2y) \, dy \]
\[ = [\sin 2y - \cos 2y]_0^{\pi/2} \]
\[ = \frac{1}{2} + \frac{1}{2} = 1. \]

Observe also that we get the same answer if we integrate in the opposite order. The area of the cross-sectional area is
\[ B(x) = \int_0^{\pi/2} \sin(x + 2y) \, dy \]
\[ = \left[ \frac{-\cos(x + 2y)}{2} \right]_0^{\pi/2} \]
\[ = \cos x - \cos(x + \pi) = \cos x. \]

Then
\[
\int \int_{R} \sin(x + 2y) \, dx \, dy = \int_0^{\pi/2} B(x) \, dx
\]
\[ = \int_0^{\pi/2} \cos x \, dx \]
\[ = [\sin x]_0^{\pi/2} = 1. \]

**Solution 8**

Polar coordinates are particularly suitable for integrating over circular regions, here the entire disc of radius \( a \). The change to polar coordinates in this integral is facilitated by the fact that \( \sqrt{x^2 + y^2} = r \). In the new variables we are faced with evaluating a double integral with integrand
\[ F(r, \theta) = h \left( 1 - \frac{r}{a} \right) r, \]
which depends on \( r \) only. This enables us to do the \( \theta \)-integration first, which contributes a factor of \( 2\pi \) to the integral. The remaining \( r \)-integrand is a polynomial, and so we encounter no problems.

In detail,
\[
\int \int_{R} h \left( 1 - \frac{\sqrt{x^2 + y^2}}{a} \right) \, dx \, dy = \int_{r=0}^{r=a} \int_{\theta=0}^{\theta=2\pi} h \left( 1 - \frac{r}{a} \right) r \, dr \, d\theta
\]
\[ = h \left( \int_0^{2\pi} d\theta \right) \times \left( \int_0^{a} \left( 1 - \frac{r}{a} \right) r \, dr \right) \]
\[ = 2\pi h \left[ \frac{1}{2} r^2 - \frac{r^3}{3} \right]_0^a \]
\[ = \frac{1}{3} \pi ha^2. \]

Recall that
\[
\int \int_{R} F(x, y) \, dx \, dy
\]
represents the volume of the figure whose base is $R$ and the volume is determined by the equation $z = F(x, y)$. To review this, see Figure 15 in Subsection 4.2. In this problem the equation

$$z = F(x, y) = h \left( 1 - \frac{\sqrt{x^2 + y^2}}{a} \right)$$

represents the surface of a cone with base $x^2 + y^2 \leq a^2$ and height $h$, so the above integral gives its volume.

![Figure 26](image)

**Solution 9**

(a) Using integration by parts twice, we have

$$\int_0^\infty e^{-xt} \cos t \, dt = \left[ e^{-xt} \sin t \right]_0^\infty + x \int_0^\infty e^{-xt} \sin t \, dt$$

$$= -xe^{-x^2} \cos t \bigg|_0^\infty - x^2 \int_0^\infty e^{-xt} \cos t \, dt$$

$$= x - x^2 \int_0^\infty e^{-xt} \cos t \, dt.$$

Therefore $(1 + x^2) \int_0^\infty e^{-xt} \cos t \, dt = x$.

Dividing through by $(1 + x^2)$ then gives the Laplace transform of $\cos t$.

(b) We substitute $u = 2t$ and obtain

$$\int_0^\infty e^{-xt} \cos 2t \, dt = \frac{1}{2} \int_0^\infty e^{-xu/2} \cos u \, du$$

$$= \frac{1}{2} \left( \frac{x/2}{x^2/4 + 1} \right)$$

(see (a))

$$= \frac{x}{4 + x^2}.$$
We employ a trigonometric identity and the previous result to obtain
\[
\int_0^\infty e^{-xt} \sin^2 t \, dt = \frac{1}{2} \int_0^\infty e^{-xt} (1 - \cos 2t) \, dt
\]
\[
= \frac{1}{2} \int_0^\infty e^{-xt} \, dt - \frac{1}{2} \int_0^\infty e^{-xt} \cos 2t \, dt
\]
\[
= \frac{1}{2} \left( \frac{1}{x} - \frac{x}{2(4 + x^2)} \right)
\]
\[
= \frac{1}{x(4 + x^2)}.
\]

**Solution 10**

This is a technique we have used before, in evaluating the integral of \(\sin x / x\).

We replace part of the integrand, here \(x^{-2}\), by an integral over a second variable, here \(t\). The resulting double integral is then easier to do for some particular reason. Here, it turns out that the integral of \(\int_0^\infty \sin^2 x e^{-xt} \, dx\) can be done in elementary terms: it is the Laplace transform of \(\sin^2 x\). In detail,

\[
\int_0^\infty \frac{\sin^2 x}{x^2} \, dx = \int_0^\infty \int_0^\infty t e^{-xt} \sin^2 x \, dx \, dt
\]
\[
= \int_0^\infty \left( \int_0^\infty e^{-xt} \sin^2 x \, dx \right) \, dt
\]
\[
= \int_0^\infty t \left( \frac{2}{t(4 + t^2)} \right) \, dt \quad \text{(see SAQ 9)}
\]
\[
= \int_0^\infty \frac{2dt}{4 + t^2}
\]
\[
= \left[ \tan^{-1} t \right]_0^\infty = \frac{\pi}{2}.
\]

Having done a number of informal calculations, you must bear in mind that they can only be justified by the sort of rigorous approach we took in the text.