THE LEBESGUE INTEGRAL

Unit 5
Definite and Indefinite Integrals
Set Book


It is essential to have this book; the course is based on it and will not make sense without it.

This unit is based on Subsections 3.2, 3.3 and 3.4 of the set book, pages 30 to 61.

Bibliography

The following book is referred to quite frequently, and is useful though not essential.


Conventions

Before starting work on this text, please read the *Guide to the Course*.

The set book is referred to as *Weir*, and the above book *Calculus*, by M. Spivak, is referred to as *Spivak*.

Acknowledgement

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Introduction

At the end of Unit 4, *The Lebesgue Integral on \( \mathbb{R} \)*, we had achieved our aim of specifying the vector space of functions, \( L^1 \), which is the domain of the Lebesgue integration operator; we now have a vector space of functions from \( \mathbb{R} \) to \( \mathbb{R} \) and a definition of the integral of these functions which is a linear transformation from \( L^1 \) to \( \mathbb{R} \).

The following questions are now pertinent.

A  How does the space \( L^1 \) of Lebesgue integrable functions we have defined relate to the Riemann integrable functions studied in Unit 2?

B  Can we continue to use the techniques of integration for the Riemann integral to compute the Lebesgue integral?

Let us look again at the specification of \( L^1 \).

![Figure 1](image_url)

We started with characteristic functions of bounded intervals \( I \) and defined the integral of such a function, \( \chi_I \) say, to be the length of \( I \). We then extended this definition to the vector space spanned by the set of all characteristic functions of bounded intervals, and we called this vector space \( S \), the set of all step functions on \( \mathbb{R} \). At the next stage we considered increasing sequences of step functions with bounded integrals, and saw that they converge almost everywhere (i.e. except possibly on null sets). We denoted by \( L^{inc} \) the set of all functions \( f \), where \( f \) is the limit almost everywhere of such a sequence of step functions. We then used the limiting process to extend the definition of the integral from functions in \( S \) to functions in \( L^{inc} \). If

\[
\{ \phi_n \} \uparrow f \text{ a.e. and } f \in L^{inc},
\]

we defined

\[
\int f = \lim \left\{ \int \phi_n \right\}.
\]

Finally, we considered the vector space \( L^1 \) spanned by \( L^{inc} \). We defined \( L^1 \) to be the set of all those functions \( f \) which can be expressed as the difference of two functions in \( L^{inc} \):

\[
f \in L^1 \text{ if there exist } g, h \in L^{inc} \text{ such that } f = g - h.
\]
The extension of the integral to $L^1$ was then achieved by defining

$$\int f = \int g - \int h.$$  

(a) Characteristic function

\[
\begin{array}{c}
\chi_{[a_1, a_2]} \\
\hline
a_1 & a_2
\end{array}
\]

\[
\int \chi_{[a_1, a_2]} = a_2 - a_1
\]

Figure 2

(b) Step function

\[
\int \phi = b_1 \int \chi_{(a_1, a_2]} + b_2 \int \chi_{(a_2, a_3]} + b_3 \int \chi_{(a_3, a_4]}
= b_1(a_2 - a_1) + b_2(a_3 - a_2) + b_3(a_4 - a_3)
\]

Figure 3

(c) Limit of increasing sequence of step functions

\[
\int f = \lim \{ \int \phi_n \}
\]

Figure 4

(d) $f \in L^1, \quad f = g - h, \quad g, h \in L^{\text{inc}}, \quad \int f = \int g - \int h.$

This summarizes what the word integral means in this course.

Let us now consider Question A. You have encountered an ‘integral’ before, when studying Riemann integration, and, while the two usages are not identical, they are
certainly related. The purpose of this unit is to study the relationship between these two concepts of integral.

The Riemann integral deals with functions on a bounded interval \([a, b]\). The Lebesgue integral, as we have defined it here, deals with functions on \(\mathbb{R}\). Now, the Lebesgue integral is versatile. We can easily proceed from our basic definition of an integral on \(\mathbb{R}\) to define one on \([a, b]\). This is done in Section 1, and we are then in a position to make a direct comparison between Lebesgue and Riemann integrals. (The Riemann integral over \(\mathbb{R}\) was defined as a limit of integrals over \([a, b]\) as \(a \to -\infty\) and \(b \to +\infty\). It was not constructed directly. As a result, we could not easily conclude precisely which functions were, and which functions were not, Riemann integrable over \(\mathbb{R}\).

We carry out this comparison in Section 2, and it is here that you will find the important theorems and proofs of this unit. The results may be summarized as follows. For a bounded function on a bounded interval, continuity a.e. is:

(a) a necessary and sufficient condition for Riemann integrability;
(b) a sufficient but not necessary condition for Lebesgue integrability.

For Riemann integrable functions, the Lebesgue and Riemann integrals have the same values.

In Section 3 we discuss some further questions of integrability. We study a very important example of a bounded function on a bounded interval that is Lebesgue integrable but is significantly different from (not a.e. equal to) any Riemann integrable function. This same example also shows that, given an increasing sequence of functions, Riemann integration does not commute with the limiting process, that is to say, the Riemann integral does not possess Property III which we demand (cf. Introductions of Units 3 and 4) of the Lebesgue integral. This convergence property lies at the heart of Lebesgue integration theory, as we shall see later. We also note in Section 3 that (Lebesgue) integrability of a function involves two essentially independent criteria: 'smoothness' and 'smallness'.

Finally, in Section 4 we consider Question B and present some comments on the evaluation of integrals. Since the Riemann and Lebesgue integrals are identical for Riemann integrable functions, the standard techniques for evaluating Riemann integrals can be used to evaluate Lebesgue integrals in all cases to which they are applicable.

# 1 Integrals on bounded intervals

In our development of the Lebesgue integral we have considered functions defined on the whole of \(\mathbb{R}\) and we introduced a convention to deal with functions defined only on some subset of \(\mathbb{R}\) (Weir: page 22). However, if we want to make comparisons between the Riemann and Lebesgue definitions of the integral, we need to incorporate in the Lebesgue theory the integral of a function defined on an interval \(I\) of \(\mathbb{R}\). There is a simple way of doing this.

When we consider the behaviour of a function \(f\) for some interval \(I\), it is as though we are wearing blinkers; we do not care about \(f\) at points not in \(I\). We can achieve the same effect by considering the product function \(f\chi_I\), where \(\chi_I\) is the characteristic function of \(I\).

In the first reading section, we use this process to introduce some further classes of functions. We shall say that \(f \in L^1(I)\) if \(g = f\chi_I \in L^1\). We shall occasionally write \(L^1(\mathbb{R})\) for \(L^1\) when we wish to emphasize that the set of functions under consideration is the set of all integrable functions with domain \(\mathbb{R}\). With this notation, we shall prove a couple of propositions which should be familiar to you.
Notes
1 Page 40, line 2. In particular, $L^1[a, b]$ consists of the functions which are integrable on $[a, b]$. (We drop a redundant pair of parentheses.)

2 Page 41, line 2. Consider the value of each side of the equation at each point of $R$.

3 Page 41, line -2. Do not look ahead to the remarks after the statement of Theorem 3.3.1 now, as they are rather difficult to follow out of context. Their import is as follows.

A function which satisfies the conditions of Theorem 3.3.1 is shown to belong to $L^{inc}$. In addition, Weir remarks that this theorem gives a necessary and sufficient condition for a function to he Riemann integrable, and so all Riemann integrable functions satisfy it. Therefore

$L^{inc}$ contains all Riemann integrable functions.

Exercise 1

Let $I$ be an interval.

(a) Show that, for any step function $\phi$, $\phi_{\chi_I}$ is also a step function.

(b) If $\{\phi_n\}$ is an increasing sequence of step functions, show that $\{\phi_n \chi_I\}$ is also an increasing sequence of step functions.

(c) Show that $\int \phi_{\chi_I} \leq \int \phi$ for any positive step function $\phi$.

(d) Show that if $f \in L^{inc}$ then $f \chi_I \in L^{inc}$.

(e) Hence show that if $J$ is another interval and $I \subset J$, then $L^1(I) \supset L^1(J)$; in particular, $L^1(I) \supset L^1(R)$.

This exercise is Weir: page 42, Exercise 7.

Solution

(a) Since $\phi$ is a step function, we can write

$$\phi = c_1\chi_{I_1} + c_2\chi_{I_2} + \ldots + c_n\chi_{I_n},$$

where the $I_j, j = 1, \ldots, n$, are bounded intervals; hence

$$\phi \chi_I = c_1\chi_{I_1}\chi_I + c_2\chi_{I_2}\chi_I + \ldots + c_n\chi_{I_n}\chi_I
= c_1\chi_{I_1 \cap I} + c_2\chi_{I_2 \cap I} + \ldots + c_n\chi_{I_n \cap I}.$$

As each $I_j \cap I$ is a bounded interval (possibly empty), it follows that $\phi \chi_I$ is also a step function.

(b) If $x \notin I$, then

$$(\phi_n \chi_I)(x) = \phi_n(x)\chi_I(x) = 0$$

for all $n$, so, in particular,

$$(\phi_{n+1} \chi_I)(x) = 0 = (\phi_n \chi_I)(x).$$

If $x \in I$, then

$$(\phi_{n+1} \chi_I)(x) = \phi_{n+1}(x)\chi_I(x) = \phi_{n+1}(x) \geq \phi_n(x) = \phi_n(x)\chi_I(x) = (\phi_n \chi_I)(x).$$

So for all $x$, $(\phi_{n+1} \chi_I)(x) \geq (\phi_n \chi_I)(x)$; that is, $\{\phi_n \chi_I\}$ is an increasing sequence of step functions.
(c) For all \( x \in \mathbb{R}, 0 \leq \phi(x), \) and
\[ 0 \leq \chi_I(x) \leq 1. \]
Therefore, for all \( x \in \mathbb{R}, \)
\[ 0 \leq \phi(x)\chi_I(x) \leq \phi(x); \]
so, as step functions,
\[ 0 \leq \phi \chi_I \leq \phi, \]
and therefore, since \( f \) is order-preserving,
\[ 0 \leq \int \phi \chi_I \leq \int \phi. \]
(d) Since \( f \in L^{\text{inc}}, \) there is an increasing sequence of step functions \( \{\phi_n\}, \) with bounded integrals, such that \( \{\phi_n\} \uparrow f \ a.e. \) Using the result of part (c) for the positive step function \( (\phi_n - \phi_1), \) we have
\[ \int (\phi_n - \phi_1) \chi_I \leq \int (\phi_n - \phi_1) \quad \text{for each } n. \]
That is,
\[ \int \phi_n \chi_I \leq \int \phi_n - \int \phi_1 + \int \phi_1 \chi_I. \]
Combining this with the results of parts (a) and (b), \( \{\phi_n \chi_I\} \) is an increasing sequence of step functions with bounded integrals; the result \( f \chi_I \in L^{\text{inc}} \) follows if we can show that
\[ \{\phi_n \chi_I\} \uparrow f \chi_I \ a.e. \]
Now, we are given that \( \phi_n(\cdot) \rightarrow f(\cdot) \) for all \( x \) outside some null set \( N. \)
For \( x \notin I, \quad \phi_n(\cdot) \chi_I(\cdot) \rightarrow f(\cdot) \chi_I(\cdot) \) trivially (each side is zero).
For \( x \in I \setminus N, \quad \phi_n(\cdot) \chi_I(\cdot) = \phi_n(\cdot) \rightarrow f(\cdot) = f(\cdot) \chi_I(\cdot). \]
Therefore \( \{\phi_n \chi_I\} \) converges to \( f \chi_I \) for all \( x \) outside the null set \( I \cap N, \) that is, almost everywhere. Therefore \( f \chi_I \in L^{\text{inc}}. \)

(e) Suppose that \( f \in L^1(I). \) Then \( f \chi_J \) is integrable, so we may write
\[ f \chi_J = g - h, \]
where \( g \) and \( h \) are in \( L^{\text{inc}}. \) Part (d) shows that \( g \chi_I \) and \( h \chi_I \) are also in \( L^{\text{inc}}, \)
and therefore \( f \chi_J \chi_I = g \chi_I - h \chi_I \) is integrable.
Now
\[ f \chi_I \chi_J = f \chi_{I \cap J} = f \chi_I \]
so \( f \in L^1(I). \) That is, \( L^1(I) \subset L^1(I). \) In particular, \( I \subset \mathbb{R}, \) so \( L^1(\mathbb{R}) \subset L^1(I). \)

Exercise 2

(a) \textit{Weir:} page 43, Exercise 8.
(b) \textit{Weir:} page 43, Exercise 10, last line. (The first part of Exercise 10 was SAQ 5 in Unit 4.)

Solution

(a) Let \( I = [a, b], J_1 = [a, c] \) and \( J_2 = [c, b] \). Then \( J_1 \subset I \) and \( J_2 \subset I \) so from part (e) of the previous exercise we know that \( L^1(J_1) \subset L^1(I) \) and \( L^1(J_2) \subset L^1(I). \) That is, if \( f \in L^1(I), \) then \( f \in L^1(J_1) \) and \( f \in L^1(J_2). \) Now apply Proposition 3.2.1 (\textit{Weir:} page 40) and the result follows.

(b) See \textit{Weir:} page 245.
Exercise 3

Use the constant function \( f : x \mapsto 1 \) to show that \( L^1(I) \neq L^1(\mathbb{R}) \), where \( I \) is any bounded interval on \( \mathbb{R} \).

Solution We use a contradiction argument. Let \( f \) be the constant function \( f : x \mapsto 1 \), and \( I \) be a bounded interval on \( \mathbb{R} \). Then \( f \in L^1(I) \) because \( f \chi_I = \chi_I \) is a step function. Suppose \( f \in L^1(\mathbb{R}) \) and \( \int f = K \). For every integer \( n, f \geq f \chi_{[0,n]} \), so we would have

\[
K = \int f \geq \int f \chi_{[0,n]} = \int \chi_{[0,n]} = n \quad \text{for every } n,
\]

and this is impossible by the Axiom of Archimedes (Weir: page 3). Hence the constant function \( f : x \mapsto 1 \) has a very important property: it belongs to \( L^1(I) \) but does not belong to \( L^1(\mathbb{R}) \), so \( L^1(I) \neq L^1(\mathbb{R}) \). We shall quote this example later in the course.

2 Riemann integrals

This section is based on Weir: Subsection 3.3, pages 44-54. Before starting to read this passage, note the following comments.

Comments

(a) Pages 44-45 are introductory.

(b) The important part of the passage is on pages 46-47. The real heart of the matter is Theorem 3.3.1. This is stated at the top of page 47, but the proof is to be found on the top half of page 46. The discussion on page 47 following Theorem 3.3.1 is important, but you may find it hard going at first. We shall be discussing these and related ideas throughout this section and the next one.

Suggestion: Read page 47 twice now and once again after studying Section 3.

(c) The material on pages 48-50 is not fundamental but indicates some concrete interpretations of Theorem 3.3.1.

(d) Pages 51-54 are optional. They contain an outline of the proof that a bounded function on a bounded interval is Riemann integrable if and only if it is continuous almost everywhere.

We shall make a number of references later on to this result. It extends the fact (which you met in Unit 2) that a bounded function on a bounded interval can be Riemann integrable over that interval, even though it is not continuous at each point of the interval.

We can now give the positive connection between continuity and integrability, in the light of our (new) knowledge of null sets and the Lebesgue integral.

You are not required to study the proof. However, we do hope that your interest is sufficiently aroused so that you read through it quickly, at least. This should give you some idea of what the proof involves.

Discussion

In the proof of Theorem 3.3.1 the task is to prove integrability for any a.e. continuous function on a bounded interval.

As a preliminary to this, consider the following special case.

Proposition The function \( f : x \mapsto x \) is in \( L^{\text{inc}}[a,b] \).

Proof On any interval \([a,b]\), \( f \) may be approximated by an increasing sequence \( \{\phi_n\} \) of step functions as suggested in the following diagrams:
Specifically,
(a) \[ \phi_n(x) = a + (k - 1)(b - a)/2^n \]
for \( a + (k - 1)(b - a)/2^n \leq x < a + k(b - a)/2^n \) \((k = 1, 2, \ldots, 2^n)\)
\[ \phi_n(b) = b. \]
Furthermore,
(b) \[ \int \phi_n = \sum_{k=1}^{2^n} \left[ a + (k - 1)(b - a)/2^n \right] (b - a)/2^n \]
\[ < \sum_{k=1}^{2^n} b(b - a)/2^n \]
\[ = b(b - a), \]
so \( \{ \int \phi_n \} \) is bounded.
(c) Now the sequence of step functions \( \{ \phi_n \} \) is increasing and, for each \( x \in [a, b] \),
\[ f(x) - \phi_n(x) < (b - a)/2^n, \]
so
\[ \phi_n(x) \to f(x) \quad \text{for each } x \in [a, b]. \]
It follows that \( f \in L^{\text{inc}}[a, b] \).

This specific example sets the pattern for a general proof. Note the following.
(a) We gave a specific formula for \( \phi_n(x) \), but we could equally well have said that
\[ \phi_n(x) = \inf \{ f(x) : x \in I_k \}, \]
where
\[ I_k = \{ x : a + (k - 1)(b - a)/2^n \leq x < a + k(b - a)/2^n \}. \]
This latter formulation fits the general case.
(b) We computed \( \int \phi_n \) explicitly, but the inequality \( \int \phi_n \leq b(b - a) \) is equally well
seen from the fact that
\[ \phi_n(x) \leq f(x) \leq b \quad \text{for all } x \in [a, b]. \]
This latter form of the argument fits the general case (though the upper bound
on \( f \) may be something other than \( b \)).
(c) We obtained the result \( \phi_n(x) \to f(x) \) from the inequality
\[ f(x) - \phi_n(x) < (b - a)/2^n, \]
which is intrinsic to the specific example being considered. In the general case this result comes from continuity, and it is
helpful to note the following two pictures.
If we take an appropriate box of size $2\varepsilon$ by $2\delta$ around a point of continuity, the function stays in the box, and therefore the infimum function is also in the box.

At a point of discontinuity this is not the case if $\varepsilon$ is too small.

If you recall the definition of the Riemann integral, you will recognize the $\int \phi_n$ in the above discussions (both specific and general) as lower sums for the function in question. Thus, the proof of Theorem 3.3.1 shows that, with the given hypothesis, the Riemann lower sums tend to the Lebesgue integral. Combine this with Weir: pages 51–54 (which is optional reading) and you have for bounded functions on bounded intervals the following implications:

\[
\begin{align*}
\text{f is} & \quad \iff \quad \text{f is Riemann integrable} \quad \iff \quad \text{f is Lebesgue integrable} \\
\text{continuous} & \quad \text{a.e.} & \quad \implies & \quad \text{Lebesgue and Riemann integrals of f are equal}
\end{align*}
\]

While you will not be required to reproduce the proofs, you should get the above implications firmly in mind. We shall refer to them a number of times throughout this unit.

Read Weir: Chapter 3, Subsection 3.3, pages 44 to 50.

Notes


2 Page 46, line 5. There is a misprint in the first printing of the set book: for $|f(x) - f(x)|$ read $|f(x) - f(p)|$. 
3 Page 46, lines -15 to -10. This appeal to area is useful in helping to see what is happening, but it is not essential. All that we need is the existence of the relevant infima in page 45, line 1, and these will exist provided \( f \) is bounded below.

4 Page 47, lines 13 to 16. See Spivak: Chapter 13 or Unit 2, The Riemann Integral.

5 Page 47, lines 21 to 23. Unit 2 uses lower and upper sums calculated from partitions of the interval concerned, which converge to a common value, to show that a function is Riemann integrable. We have pointed out that the integrals of the step functions constructed in the proof of Theorem 3.3.1 can be interpreted as lower sums. Weir is making the observation that, because sup \( f = - \inf (-f) \), we can also get upper sums.

6 Page 47, lines -13 and -12. 'Improper' integrals will be discussed in Section 3 and studied in some detail in the unit on the Monotone Convergence Theorem.

7 Page 48, lines 8 and 9. This is Spivak: page 100, Theorem 2. It was also proved in M203, Analysis Block A, Unit 4 and used in Unit 2 of this course.

8 Page 48, Fig. 23. Here \( f(p - 0) \) is what Spivak calls \( \lim_{x \to p^-} f(x) \) and \( f(p + 0) \) is \( \lim_{x \to p^+} f(x) \).

9 Page 49, lines -9 to -5. See the Rational Density Theorem, part (b) of Proposition T.1.1.3 of Unit 1, The Real Numbers. This allows us to associate a rational number with each 'jump' interval. The choice of rational numbers is not unique, of course, but, having made a choice, there is a one-one correspondence between these rational numbers (which are countable) and the discontinuities. Thus the set of discontinuities is countable and therefore null.

10 Page 49, Corollary 2. If \( f \) is monotone on \( [a, b] \), then \( f \) is bounded on \( [a, b] \), the bounds being \( f(a) \) and \( f(b) \). This fact, together with the fact that the discontinuities of \( f \) form a null set, completes the set of conditions for Theorem 3.1.1. (See Exercise 8.)

Exercise 4

Does the function \( f \) defined by

\[
f : x \mapsto \begin{cases} (-1)^n & \text{if } x \in (1/(n+1), 1/n], \quad n \in \mathbb{Z}^+, \\ 0 & \text{otherwise}, \end{cases}
\]

belong to \( L^1 \)?

Solution: The function \( f \) satisfies the conditions of Theorem 3.3.1; \( f \) vanishes outside \([0, 1]\), \( f \) is bounded below by \(-1\) and above by \(+1\), and \( f \) is discontinuous on the null set \( \{1/n : n \in \mathbb{Z}^+\} \cup \{0\} \). Therefore \( f \in L^1 \).

Exercise 5

Let \( f \) be defined on \([0, 1]\) by

\[
f : x \mapsto \begin{cases} 1/q & \text{if } x \text{ is rational and } x = p/q \text{ (in lowest terms)}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}
\]

(a) Determine the set of discontinuities for \( f \).

(b) Show that \( f \) is Riemann integrable.

(c) Find the value of the Riemann integral

\[
(R) \int_0^1 f(x) \, dx.
\]
Solution

(a) If \( r \) is rational, then \( f(r) \neq 0 \). However, every open interval about \( r \) contains an irrational \( x \), and \( f(x) = 0 \). So \( f \) is not continuous at \( r \).

Suppose \( x \) is irrational and \( \varepsilon > 0 \) is given. There is a finite set of rationals \( p/q \) in \([0, 1]\) such that \( 1/q \geq \varepsilon \). Let \( r \) be the closest of these rationals to \( x \), and set \( \delta = |x - r| > 0 \). Now, if \( |x - y| < \delta \), then \( |f(y)| < \varepsilon \) whether \( y \) is rational or irrational; and so, since \( f(x) = 0 \), \( f \) is continuous at \( x \). The set of discontinuities for \( f \) is therefore the set of rational points in \([0, 1]\). (See Spivak: pages 79–80.)

(b) Since the rationals form a null set, it follows from part (a) that \( f \) is Riemann integrable.

(c) Since \( f = 0 \) almost everywhere, it follows by Weir: Theorem 3.2.3 that the Lebesgue integral of \( f \) is 0. However, since \( f \) is Riemann integrable, its Riemann and Lebesgue integrals are equal, so

\[
(R) \int_0^1 f(x) \, dx = 0.
\]

Note carefully the reasoning here; Theorem 3.2.3 does not apply to Riemann integrals.

Exercise 6

Let \( f \) be the function given by

\[
f : x \rightarrow \begin{cases} 
  n(n + 1)x + n + 1 & \text{if } x \in [1 - (1/n), 1 - 1/(n + 1)), \quad n \in \mathbb{Z}^+, \\
  0 & \text{otherwise.}
\end{cases}
\]

Figure 9
Let \( \theta \) be a real number, \( 0 < \theta < 1 \).

(a) Show that \( f \in L^1[0, \theta] \).

(b) Show that if \( 1 - (1/n) \leq \theta < 1 - 1/(n+1) \), then
\[
\int_0^\theta f \geq n - 1.
\]

(c) Hence show that \( f \notin L^1[0, 1] \).

**Solution**

(a) The function \( f \) is monotone on \([0, \theta]\) and therefore integrable on \([0, \theta]\) (Corollary 3.3.2, Weir: page 49).

(b) Given two intervals \( I \) and \( J \) such that \( I \subsetneq J \), and a positive function \( f \in L^1(I) \), we know from Exercise 1 that \( f \in L^1(J) \), and since
\[
f_{x_I} \geq f_{x_J}
\]
we have
\[
\int_0^\theta f \geq \int f_{x_J}.
\]

Now take our given function as the positive function \( f \), and consider the intervals \( I = [0, \theta], J = [0, 1 - 1/n] \). We may deduce that
\[
\int_0^{1 - 1/n} f = \sum_{r=1}^{n-1} \int_{1 - 1/r}^{1 - 1/(r+1)} f.
\]

From repeated application of Proposition 3.2.1 (Weir: page 40), and the use of Exercise 1, part (e), which tells us that \( f \) is integrable over the subintervals of the form \([1 - 1/r, 1 - 1/(r+1)]\), we have
\[
\int_{1 - 1/r}^{1 - 1/(r+1)} f \geq r(r+1) f_{x_I}.
\]

Since \( f \) is increasing, we have
\[
f(x) \geq f(1 - 1/r) = r(r+1) f_{x_I} \quad \text{if} \quad x \in \left[1 - \frac{1}{r}, 1 - \frac{1}{r+1}\right],
\]
and so
\[
\int_{1 - 1/r}^{1 - 1/(r+1)} f \geq r(r+1) \int_{1 - 1/r}^{1 - 1/(r+1)} f_{x_I} = 1.
\]

Therefore
\[
\int_0^\theta f \geq \sum_{r=1}^{n-1} 1 = n - 1.
\]

(c) Suppose \( f \) is integrable on \([0, 1]\); then
\[
\int_0^1 f = k
\]
for some real number \( k \). On the other hand, we have
\[
\int_0^1 f \geq \int_0^{1 - 1/n} f \geq n - 1
\]
for every integer \( n \geq 1 \). That is,
\[
k \geq n - 1 \quad \text{for all} \quad n \geq 1,
\]
which is clearly impossible, by the Axiom of Archimedes. It follows that \( f \) is not integrable on \([0, 1]\).
3 Integrability

In this section we shall not introduce any additional material from Weir. Rather, our purpose here is to summarize and consolidate what we have learned so far about the concept of integrable function.

The Riemann definition of an integral can be formulated only for a bounded function on a bounded interval. (The domain must be bounded because it must be partitioned into a finite set of subintervals each with finite length. The function must be bounded because we use infima to form lower sums and suprema to form upper sums.)

By contrast, the Lebesgue definition has no built-in boundedness requirement. In many unbounded cases we do not get an integrable function, but at least we can set up the machinery and investigate.

Figure 10

To put it another way, if figures like those above have ‘finite area’, the Lebesgue definition of an integral will give these areas directly. There has been some mention of ‘improper’ integrals (Weir: page 47, line −12). In the Riemann theory, recall, this phrase refers to integrals giving areas of unbounded figures. In the Lebesgue theory, there is no ‘impropriety’ involved; such integrals are covered by the definition.

If a function \( f \) is Riemann integrable over the whole real line in the sense discussed in Unit 2, then \( f \) may or may not be Lebesgue integrable over \( \mathbb{R} \). This is in contrast to the fact that if a function is Riemann integrable over a closed and bounded interval, it is Lebesgue integrable over that interval.

The following informal description of integrability may prove helpful. Roughly speaking, for a function \( f \) to be integrable (in any sense) it must be smooth enough and small enough. Smallness has to do with the values of the function: does it take infinite values, and if so, on what sets; does it approach zero at infinity fast enough? Smoothness has to do with turning points, oscillations and discontinuities.

If \( f \) is bounded and restricted to a bounded interval \( I \), the smallness criterion is automatically satisfied for both Riemann and Lebesgue theories. Smoothness is the only significant problem in this case. The Riemann smoothness criterion is precisely continuity almost everywhere. As the example below shows, the Lebesgue smoothness criterion is not so stringent.

In general terms, the Lebesgue smoothness criterion is that \( f \) be the difference of two functions, each of which is the limit almost everywhere of an increasing sequence of step functions. As a working tool, this is not particularly satisfactory, and our advice at the present stage in the course is not to worry unduly about the Lebesgue smoothness criterion. Unhappily, not every function is smooth enough, but it really is quite a task to construct one that is too ‘rough’. A much clearer picture of this situation will emerge in the unit on Lebesgue Measure.
Practical methods for testing specific functions for smallness will be described in the units on the Monotone and Dominated Convergence Theorems. However, some informal comments on the subject may be useful now.

It is an essential feature of the Lebesgue theory that the smallness criterion be applied at the $L^{\infty}$ level. That is, for $f$ to be in $L^1$ it must be the difference of two functions in $L^{\infty}$ each of which passes the smallness test in its own right. Now, it suffices to consider non-negative functions, because once we get $\{\phi_n\} \uparrow f$ we can always turn our attention to $f - \phi_1$. Thus, the picture of the Lebesgue smallness criterion is obtained by answering the question:

**If $f$ is non-negative and smooth enough, what smallness condition must it satisfy in order to be in $L^{\infty}$?**

Suppose $f \geq 0$ and $\{\phi_n\}$ is an increasing sequence of step functions. By definition, $f \in L^{\infty}$ provided:

(a) $\phi_n \rightarrow f$ a.e.;
(b) $\{f \phi_n\}$ is bounded above.

Now, let us separate these two conditions. Essentially, (a) is the smoothness criterion and (b) is the smallness criterion.

This can be expressed a little more precisely as follows. In proving integrability, we can say that if (a) holds, then (b) guarantees that $f$ is small enough.

In proving non-integrability it is sufficient to show that either (a) or (b) is violated. For example, (b) is violated if we can find an increasing sequence of step functions $\{\phi_n\}$ with $\phi_n \leq f$ and $\{f \phi_n\}$ unbounded.

This last assertion follows by the argument given in the solution to Exercise 3. If $f$ were integrable, we would have $\int f \geq \int \phi_n$ for every $n$, and if $\{f \phi_n\}$ is unbounded this contradicts the Axiom of Archimedes.

**Example 1 (Important)** Let $\{I_n\}$ be a sequence of open intervals, each contained in $(0,1)$ such that

$$\bigcup_{n=1}^{\infty} I_n \supseteq \mathbb{Q} \cap (0,1)$$

and such that the sum of the lengths of the $I_n$ is less than or equal to $\frac{1}{2}$. (Such a sequence exists because the set $\mathbb{Q}$ of rationals is a null set.)

Let

$$S_k = \bigcup_{n=1}^{k} I_n, \quad S = \bigcup_{n=1}^{\infty} I_n, \quad \text{and} \quad K = [0,1] \setminus S.$$

(a) Show that $\chi_S \in L^{\infty}[0,1]$.
(b) Show that $\chi_S$ is discontinuous at each point of $K$.
(c) Show that $K$ is not a null set; hence $\chi_S$ is not Riemann integrable.
(d) Show, further, that $\chi_S$ is not a.e. equal to any Riemann integrable function.

**Solution**

(a) For each $k$, $\chi_{S_k}$ is a step function and $\chi_{S_k} \leq \sum_{n=1}^{k} x_{I_n}$; so

$$\int \chi_{S_k} \leq \sum_{n=1}^{k} \int x_{I_n} = \sum_{n=1}^{k} \ell(I_n) \leq \sum_{n=1}^{\infty} \ell(I_n) \leq \frac{1}{2}.$$

Now, $\{x_{S_k}\}$ is increasing, and for each $x$

$$\lim_{k \to \infty} x_{S_k}(x) = x_S(x);$$

so $x_S$ satisfies the conditions defining $L^{\infty}$.
(b) Let \( x \in K \) and let \( I \) be an open interval containing \( x \). The Rational Density Theorem (Proposition T.1.1.3(b), Unit 1) ensures that \( I \) contains a rational \( r \), and \( r \in S \) because \( S \) covers the rationals. Now, \( \chi_S(x) = 0 \) but \( \chi_S(r) = 1 \); so \( \chi_S \) is discontinuous at \( x \).

(c) Suppose \( K \) were a null set; then, given \( \varepsilon > 0 \), there would be a sequence \( \{J_m\} \) of open intervals such that
\[
\bigcup_{m=1}^{\infty} J_m \supset K \quad \text{and} \quad \sum_{m=1}^{\infty} \ell(J_m) < \varepsilon.
\]
Letting
\[T = \bigcup_{m=1}^{\infty} J_m,
\]
we would now have
\[
\chi_{[0,1]} \leq \chi_S + \chi_T \leq \sum_{n=1}^{\infty} \chi_{I_n} + \sum_{m=1}^{\infty} \chi_{J_m}.
\]
Using the same sort of argument as in part (a), we can see that \( \sum_{n=1}^{\infty} \chi_{I_n} \) and \( \sum_{m=1}^{\infty} \chi_{J_m} \) converge a.e. and belong to \( L^1 \) so that
\[
1 = \int \chi_{[0,1]} \leq \sum_{n=1}^{\infty} \int \chi_{I_n} + \sum_{m=1}^{\infty} \int \chi_{J_m} = \sum_{n=1}^{\infty} \ell(I_n) + \sum_{m=1}^{\infty} \ell(J_m) < \frac{1}{2} + \varepsilon.
\]
Clearly, for \( \varepsilon < \frac{1}{2} \) this is impossible, and we may deduce that \( K \) is not a null set.

Combining parts (b) and (c), it follows that \( \chi_S \) is not continuous a.e. and therefore is not Riemann integrable.

(d) Let \( N \) be a null set and let \( f = \chi_S \) on \([0,1]\setminus N\). Without specifying what values \( f \) has on \( N \), we propose to show that \( f \) is not Riemann integrable. Specifically, by Weir: Proposition 2.2.3, \( K \setminus N \) is not a null set; we shall show that \( f \) is discontinuous on \( K \setminus N \). Let \( x \in K \setminus N \) and let \( I \) be an open interval containing \( x \). We must find \( y \in I \) such that \( |f(x) - f(y)| \) is not small. As in the proof of part (b), \( I \) must intersect \( S \) (so that \( I \cap S \) is not empty); but we now note that \( I \cap S \) is open and so is not null (since it contains an open interval). Since \( N \) is null and \( I \cap S \) is not, there is a point \( y \) such that
\[
y \in (I \cap S) \setminus N.
\]
We now have
\[
f(x) = \chi_S(x) = 0 \quad \text{and} \quad f(y) = \chi_S(y) = 1,
\]
and \( f \) is discontinuous at \( x \). □

Comment

This example contains some highly significant results that deserve further comment. Clearly, it shows that the Lebesgue smoothness criterion is less stringent than the Riemann one. The significance of part (d) is that Lebesgue smoothness is significantly less restrictive than Riemann smoothness in that \( \chi_S \) cannot be converted into a Riemann integrable function by recourse to Weir: Theorem 3.2.3.
Reasonable questions might be:

(a) Why do we want less restrictive smoothness requirements?
(b) Who wants to integrate functions like \( \chi_S \) anyway?

Part (a) of the example gives a definitive answer to the first question. Not only is \( \chi_S \) in \( L^{1_{\infty}} \), but the approximating step functions \( \chi_{S_k} \) are Riemann integrable. Thus we have shown that the Riemann smoothness criterion is not preserved under the limit operation. The reason for this trouble is that the Riemann smoothness criterion is tied to continuity, and continuity is frequently lost in the pointwise convergence process.

To see this, consider the sequence of functions given by \( f_n : x \mapsto x^n \) (\( 0 \leq x \leq 1 \)). Here each \( f_n \) is continuous on \([0, 1]\), but

\[
\lim_{n \to \infty} f_n(x) = 0 \quad \text{for} \quad x \in [0, 1)
\]

and

\[
\lim_{n \to \infty} f_n(1) = 1.
\]

The limit function has a jump discontinuity at 1. In the example, each \( \chi_{S_k} \) has only a finite number of discontinuities, but the limit function \( \chi_S \) sprouts a whole non-null set full of new ones.

Regarding the second question we can say the following. Many applications of mathematics in science force us to consider functions and their integrals which are not Riemann integrable. They may have too many singularities, oscillate too rapidly, not vanish rapidly enough at infinity, or whatever. Such functions appear in Fourier transform theory, in quantum mechanics, in chaos theory, in electronics, and in many other subjects.

Fortunately, most of the functions encountered in applications can be written as limits of series of step functions. From what we now know, we are saying that the functions are in \( L^1 \). As the construction of \( L^1 \) begins with \( \chi_S \), this is a strong motivation for studying step functions.

In addition, the applications mentioned above often require consideration of limits of functions and the integrals of such limits. Here, a notion of limit is required which preserves smoothness when the functions of the sequence are smooth, yet can handle general \( L^1 \) functions. This is a characteristic property of limits almost everywhere, once we have incorporated the limit condition of step (c) in our construction of the integral. In a sense, then, step (c) is the real key to developing a useful extension of Riemann integration.

We now list some terminology to cover usage of the word integrable henceforth.

**Definition T.4.3**

- **Integrable** means Lebesgue integrable on \( \mathbb{R} \).
- **Integrable on \( I \)** means Lebesgue integrable on \( I \).
- **Riemann integrable** means integrable by the Riemann definition (i.e. bounded and continuous a.e. on a bounded interval).

**Exercise 7**

Show that \( f : x \mapsto x \) is not integrable.

**Solution** By Theorem 3.2.2(iii) (Weir: page 37) it suffices to show that \( |f| \notin L^1 \).

Suppose \( |f| \in L^1 \) with \( \int |f| = K \). Define a sequence \( \{ \phi_n \} \) as follows:

\[
\phi_n(x) = \begin{cases} 
  k & \text{if } k \leq |x| < k + 1 \quad (k = 0, 1, 2, \ldots, n), \\
  0 & \text{if } |x| \geq n + 1.
\end{cases}
\]
The following diagram shows $|f|$ and $\phi_n$.

\[
K = \int |f| \geq \int \phi_n = 2 \sum_{k=1}^{n} k = n(n + 1)
\]

for every $n$; but this is impossible by the Axiom of Archimedes.

Note that $\{\phi_n\}$ does not converge to $|f|$ here. To prove integrability, we must use a sequence of step functions converging a.e. to the function concerned. To prove non-integrability, this is not required. The above argument suffices.

**Comment**

The strategy adopted in this solution should be noted carefully. We showed that $f \notin L^1$ by establishing that the existence of $\int |f|$ would conflict with the Axiom of Archimedes. This technique is often useful.

**Exercise 8**

Show that $f : x \mapsto 1/x$ is not integrable on $(0, 1]$.

**Solution** Define step functions $\phi_n$ as follows:

\[
\phi_n(x) = \begin{cases} 
  n/k & \text{if } (k - 1)/n < x \leq k/n \\
  0 & \text{if } x \notin (0, 1].
\end{cases} \quad (k = 1, 2, \ldots, n),
\]

Then, $\phi_n \leq f$ and

\[
\int \phi_n = \sum_{k=1}^{n} (n/k) \times (1/n) = \sum_{k=1}^{n} 1/k.
\]

Since $\sum 1/k$ is a divergent series, $\{\int \phi_n\}$ is unbounded, and the result follows.

(This exercise shows that Corollary 2, Weir: page 49, does not hold for a monotonic function defined on an interval of the form $(a, b]$. In general, such a function need not be bounded.)

We suggest that you now re-read Weir: page 47.
4 Evaluation of integrals

The normal calculus technique for evaluating
\[ \int_0^1 x^2 \, dx \]
is to write
\[ \int_0^1 x^2 \, dx = \left[ \frac{1}{3} x^3 \right]_0^1 = \frac{1}{3}. \]

Unfortunately, because of the intermediate step here, there is a widespread tendency to refer to \( \frac{1}{3} x^3 \) as the 'integral of \( x^2 \)'. Some people feel a little guilty about this and compromise by saying \( \frac{1}{3} x^3 \) is the 'indefinite integral of \( x^2 \'). We want to suppress both these usages here.

The reason \( \frac{1}{3} x^3 \) is the appropriate choice in the above example is that the derived function of \( x \mapsto \frac{1}{3} x^3 \) is \( x \mapsto x^2 \). The term used by Spivak to describe the reverse relationship is primitive. The above example is correct because \( x \mapsto \frac{1}{3} x^3 \) is a primitive of \( x \mapsto x^2 \), and the Fundamental Theorem of Calculus asserts that the integral of a continuous function \( f \) is an appropriate difference of values of a primitive of \( f \). We want to stress that with our use of the word integral, the integral of \( f \) is a number. By contrast, a primitive of \( f \) is a function; if you want to refer to it, please use some name that does not involve the word integral.

Weir: Subsection 3.4, pages 54–61, develops the standard techniques for evaluating integrals of continuous functions on bounded intervals. These are: the Fundamental Theorem of Calculus; Integration by Parts; and Integration by Substitution.

Since the discussion is limited to continuous functions on bounded intervals, it is strictly Riemann integration theory and presents nothing not found in Spivak. However, since the Lebesgue and Riemann integrals are identical where the Riemann integral is defined, everything in this section applies to Lebesgue integrals too. We invite you to read these pages, but this material will not be assessed now. In Unit 8, The Monotone Convergence Theorem, we shall study formally the extension of these techniques to non-Riemann situations, primarily those encountered when we remove the boundedness restrictions.

In Subsection 3.5, pages 63–69, Weir discusses some of the things that can happen if we drop the rather stringent smoothness restriction of continuity everywhere. If \( f \) has a jump discontinuity at \( x_0 \), then the function \( F \) defined by
\[ F(x) = \int_0^x f(t) \, dt \]
is not differentiable at \( x_0 \), and the best we can hope for is 'a.e. primitives'. However, in the following diagram we have \( F_1' = F_2' = f \) a.e.

So 'a.e. primitives' are not unique up to an additive constant, and validity of the Fundamental Theorem of Calculus formula depends on which primitive you use. In the above picture \( F_1 \) gives the right answer, and a snap judgement might be 'Oh, this is easy, just use the continuous one'. If you believe that and want to be disillusioned, study Weir: page 69, Exercise 10.
Weir says on page 43 that Subsection 3.5 should be skipped by the beginner. It is certainly not part of this course; but if we have whetted your appetite, go ahead!

5 Summary of the text

In this unit we considered the following problems.

A How does the space $L^1$ of Lebesgue integrable functions we have defined relate to the Riemann integrable functions?

B Can we continue to use the techniques of integration for the Riemann integral to compute the Lebesgue integral?

We gave some simple sufficient criteria for Riemann integrability. These criteria apply to functions of the form $f_{X,t}$, where $I$ is a closed, bounded interval in $\mathbb{R}$; they are listed below under Integrability Conditions. Using these criteria we were able to answer Problem A:

Riemann integrability implies Lebesgue integrability.

The simple answer to Problem B is 'Yes': all techniques from Riemann integration can be carried over to the Lebesgue context since the (proper) Riemann integral of a function (when it exists) is equal to the Lebesgue integral of the function. These techniques are listed below under Integration Techniques.

The simple approach to the Lebesgue integral on $\mathbb{R}$ that we have adopted in this course can be extended to higher dimensions. In the next two units we shall consider this extension.

In the following subsections references to the set book or this text are preceded by W or T respectively. For example, a reference to page 9 of the set book is indicated by [W9] and to Section 3 of this text by [T3].

5.1 Notation

\[
\int_{-\infty}^{\infty} f(x) \, dx \quad \text{[W39]}
\]

\[
\int_{-\infty}^{b} f(x) \, dx \quad \text{[W40]}
\]

\[
\int_{a}^{b} f \, dx \quad \text{[W47]}
\]

\[
L^1(\mathbb{R}) \quad \text{[T1], [W47]}
\]

\[
L^1(I) \quad \text{[T1]}
\]

\[
L^1[a,b] \quad \text{[W40]}
\]

5.2 Glossary

bounded [W46]
continuous at a point [W46]
continuous from the left [W48]
continuous from the right [W48]
continuous on an interval $I$ [W47]
increasing [W48]
integrable [T3]
integrable on $I$ [W40], [T3]
integral of $f$ on $I$ [W39]
left-hand derivative [W55]
monotone [W48]
Riemann integrable [T3]
right-hand derivative [W55]
5.3 Results

Proposition 3.2.1 [W40] and [W43] \[ \int_a^b f = \int_a^c f + \int_c^b f \]

Proposition 3.2.2 (The Mean Value Theorem for Integrals) [W41] If, for each \( x \in [a, b] \), \( m \leq f(x) \leq M \), then
\[ m(b - a) \leq \int_a^b f \leq M(b - a). \]

Note: The following are sufficient conditions for a function \( f \) to be integrable (i.e. integrability conditions).

Theorem 3.3.1 [W47] Let \( f \) be a function which vanishes outside the interval \([a, b]\). If \( f \) is bounded and if the points of discontinuity of \( f \) form a null set, then \( f \in L^1 \).

Theorem 3.3.1’ [W47] If \( f \) is bounded on \([a, b]\) and if the points of discontinuity of \( f \) on \([a, b]\) form a null set, then \( f \in L^1[a, b] \).

Note: The theorems above give the following necessary and sufficient conditions for \( f \) to be Riemann integrable.

Corollary 3.3.1 [W48] If \( f \) is continuous on \([a, b]\), then \( f \) is integrable on \([a, b]\).

Corollary 3.3.2 [W49] If \( f \) is monotone on \([a, b]\), then \( f \) is integrable on \([a, b]\).

Note: The following is a sufficient condition that a function \( f \) not be integrable.

If \( \{\phi_n\} \) is an increasing sequence of (step) functions such that \( f \geq \phi_n \) for each \( n \) and \( \{\int \phi_n\} \) is unbounded, then \( f \notin L^1 \).

This condition is often useful in conjunction with the property that \( f \in L^1 \) implies \(|f| \in L^1 \).

Note: The following are some useful integration techniques.

Theorem 3.4.2 (The Fundamental Theorem of Calculus) [W57] If the function \( F \) has the continuous derivative \( f \) on the closed interval \([a, b]\), then
\[ \int_a^b f(x) \, dx = F(b) - F(a). \]

Proposition 3.4.1 (Integration by Parts) [W58] If the functions \( F, G \) have continuous derivatives \( f, g \) respectively on \([a, b]\), then
\[ \int_a^b Fg = [FG]_a^b - \int_a^b fG. \]

Proposition 3.4.2 (Integration by Substitution) [W59] Suppose that \( G \) has a positive continuous derivative on the closed interval \([c, d]\) and write \( a = G(c), b = G(d) \). Then any function \( f \) which is continuous on the closed interval \([a, b]\) satisfies
\[ \int_a^b f(x) \, dx = \int_c^d f(G(t))G'(t) \, dt. \]
6 Self-Assessment Questions

6.1 Integrals on subsets of \( \mathbb{R} \)

SAQ 1
Let \( D \) be any subset of \( \mathbb{R} \) and define \( L^1(D) \) to be the class of functions \( f \) for which \( f \chi_D \in L^1(\mathbb{R}) \). Show that if \( D \) is a null set then every function \( f \) with domain \( \mathbb{R} \) is in \( L^1(D) \).

SAQ 2
Show that \( L^1[0,1] \) is properly contained in \( L^1[\theta,1] \), for \( 0 < \theta < 1 \). Hint: Use Exercise 8 of the text. Indicate how this result can be used to show that if the closed, bounded interval \( I \) is properly contained in the closed, bounded interval \( J \) then \( L^1(J) \) is properly contained in \( L^1(I) \).

SAQ 3
Verify that when \( I \) and \( J \) are two closed, bounded intervals
\[
L^1(I \cap J) \supseteq L^1(I) \cup L^1(J) \supseteq L^1(I \cup J).
\]
What happens when \( I \) and \( J \) are disjoint?

6.2 Integrability conditions

SAQ 4
Determine the integrability, or otherwise, giving reasons, of the following functions.

(a) \( f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational}, \\ |p| & \text{if } x = \frac{p}{q}, \text{in lowest terms}. \end{cases} \)

(This function is even more amusing to 'draw' than \( \chi_{\mathbb{Q}} \))

(b) \( f(x) = (-1)^{n+1}/n \) if \( x \in [n-1,n), \ n \in \mathbb{Z}^+ \), 0 if \( x < 0 \).

Hint: Apply Theorem 3.2.2(iii), Weir: page 37.

(c) \( f(x) = \begin{cases} x & \text{if } x \in (0,1] \text{ and the ternary expansion of} \\ 0 & \text{otherwise.} \end{cases} \)


SAQ 5
(a) Show that \( f(x) = e^{-x} \) is integrable on \([0,\infty)\), using the construction of Subsection 4.2. Hint: Define the step functions
\[
\phi_n(x) = \begin{cases} e^{-k/2^n} & \text{if } (k-1)/2^n \leq x < k/2^n \quad (k = 1, 2, \ldots, n2^n), \\ 0 & \text{otherwise}. \end{cases}
\]

(b) Why can we not integrate \( f \) using the Fundamental Theorem of Calculus?

Solutions to Self-Assessment Questions

Solution 1
Let \( f \) be any function with domain \( \mathbb{R} \). Then \( f \chi_D \) differs from the zero function only on points in \( D \), that is, on some null set. Hence \( f \chi_D \) is the zero function almost everywhere and is therefore integrable. So \( f \in L^1(D) \).
Solution 2

From Exercise 1, part (c), \( L^1[0, 1] \subset L^1[\theta, 1] \). Let \( f \) be the function \( f : t \mapsto 1/t \).
From Exercise 8, \( f \notin L^1(0, 1) \), but \( f \) is continuous on \([\theta, 1] \), so that \( f \in L^1[\theta, 1] \), using Corollary 3.3.1.

In general, if the closed, bounded interval \( I = [a, b] \) is properly contained in \( J = [a, c] \), that is if \( a < c < b \), we define
\[
f : t \mapsto \begin{cases} 1/(t-a) & \text{if } x \in (a, d], \\ 0 & \text{otherwise}, \end{cases}
\]
to illustrate the proper inclusion \( L^1(J) \subset L^1(I) \).

Solution 3

These inclusions follow from Exercise 1 and the inclusions
\[
I \cap J \subset I, \quad I \cap J \subset J, \quad I \subset I \cup J, \quad J \subset I \cup J.
\]
If \( I \cap J = \emptyset \), then \( L^1(I \cap J) \) is the set of all functions with domain \( \mathbb{R} \) (see SAQ 1).

Solution 4

(a) This function is integrable, since \( f \) is zero almost everywhere, and \( \int f = 0 \).

Note: \( f \) is not Riemann integrable.

(b) From Theorem 3.2.2(iii),
\[
\text{if } f \in L^1, \text{ then } |f| \in L^1.
\]

Since \( L^1 \subset L^1(I) \) for any interval \( I \),
\[
\text{if } |f| \in L^1, \text{ then } |f| \in L^1[0, n],
\]
where \( n \) is a strictly positive integer.

Now
\[
\int_0^n |f| = 1 + \frac{1}{2} + \ldots + \frac{1}{n},
\]
and, since \( |f| \) is positive,
\[
\int |f| \geq \int_0^n |f| \geq \sum_{r=1}^n \frac{1}{r}.
\]

This gives a contradiction because \( \{\sum_{r=1}^n 1/r : n \in \mathbb{Z}^+\} \) is not bounded, so \( f \notin L^1 \).

(c) The Cantor Set is a null set (see Unit 1, The Real Numbers, Subsection 2.3) so \( f \) is zero almost everywhere; therefore \( f \in L^1 \) and \( \int f = 0 \).

Solution 5

(a) The sequence \( \{\phi_n\} \) is increasing and
\[
f(x) - \phi_n(x) \leq e^{-(k-1)/2^n} - e^{-k/2^n} = e^{-k/2^n}(e^{1/2^n} - 1) \leq e^{-1/2^n}(e^{1/2^n} - 1)
\]
for \( 0 \leq x < n \); so \( \phi_n(x) \rightarrow f(x) \) for every \( x \in [0, \infty) \). Furthermore,
\[
\int \phi_n = \sum_{k=1}^{n2^n} e^{-k/2^n} \times (1/2^n).
\]

This is a partial sum of a geometric series
\[
\sum_{k=0}^{\infty} ar^k
\]
The sum of the entire series is \( a/(1 - r) \), and the partial sums are less than this; so
\[
\int \phi_n < \frac{a}{1 - r} = \frac{e^{-1/2^n}}{2^n(1 - e^{-1/2^n})} = \frac{1/2^n}{e^{1/2^n} - 1}.
\]

Setting \( y = 1/2^n \), we have
\[
\lim_{n \to \infty} \frac{1/2^n}{e^{1/2^n} - 1} = \lim_{y \to 0} \frac{y}{e^y - 1} = \lim_{y \to 0} \frac{1}{e^y} = 1,
\]
differentiating numerator and denominator in the next to last step (by l'Hôpital's Rule, Spivak: page 179). Thus \( \{\int \phi_n\} \) is bounded above by the least upper bound of the given convergent sequence. This establishes the result.

Note: There is a well-known formula for a partial sum of a geometric series, and using this we could have computed \( \int \phi_n \) exactly. From this we could have found the value of \( \int f \) (by coincidence, this is 1). However, the procedure we used above simplifies the computations somewhat; more importantly, it illustrates that a proof of integrability need not involve finding the value of the integral.

(b) The Fundamental Theorem of Calculus does not apply to this case, since the interval is not bounded. We could use it to deduce that \( f \) is integrable on \([0, \alpha]\) for each \( \alpha \), but this is not sufficient to show that \( f \) is integrable on \([0, \infty)\).