THE LEBESGUE INTEGRAL

Unit 6
The Lebesgue Integral
on $\mathbb{R}^k$
THE LEBESGUE INTEGRAL

Unit 6
The Lebesgue Integral on $\mathbb{R}^k$

Prepared by the Course Team
Set Book


It is essential to have this book; the course is based on it and will not make sense without it.

This unit is based on Subsections 4.1 and 4.2 of the set book, pages 70–81.

Bibliography

The following book is referred to quite frequently, and is useful though not essential.


Conventions

Before starting work on this text, please read the *Guide to the Course*.

The set book is referred to as *Weir*, and the above book *Calculus*, by M. Spivak, is referred to as *Spivak*.
## Contents

**Introduction** 4

1. **Step functions on** $\mathbb{R}^k$ 5
  1.1 Functions on $\mathbb{R}^k$ 5
  1.2 Intervals and step functions in $\mathbb{R}^k$ 8
  1.3 The integral for step functions on $\mathbb{R}^k$ 13

2. **The integral on** $\mathbb{R}^k$ 15

3. **The definite integral** 17

4. **Summary of the text** 19
   4.1 Notation 20
   4.2 Glossary 20
   4.3 Results 20

5. **Self-Assessment Questions** 21
   5.1 Step functions on $\mathbb{R}^k$ 21
   5.2 The integral on $\mathbb{R}^k$ 22
   Solutions to Self-Assessment Questions 23
Introduction

We have now completed the development of the definition of the Lebesgue integral of a function \( f : \mathbb{R} \rightarrow \mathbb{R} \). Before going on to investigate the properties of this integral, we shall generalize our results by defining the integral of a function of the form \( f : \mathbb{R}^k \rightarrow \mathbb{R} \).

Here \( \mathbb{R}^k \), where \( k \) is a positive integer, denotes \( k \)-dimensional Euclidean space; a point \( z \) in \( \mathbb{R}^k \) is a \( k \)-tuple \( (x_1, x_2, \ldots, x_k) \), where \( x_1, x_2, \ldots, x_k \) are real numbers. \( \mathbb{R}^1 \) (or \( \mathbb{R} \)) is the real line, \( \mathbb{R}^2 \) is the plane, and \( \mathbb{R}^3 \) is the familiar space of three dimensions.

The four steps in our construction of the Lebesgue integral on \( \mathbb{R} \) can be used almost verbatim to define the Lebesgue integral on \( \mathbb{R}^k \). To see this, we list the four steps again and examine where we have made essential use of the fact that the functions had domain \( \mathbb{R} \) (and not \( \mathbb{R}^k \)).

(a) Characteristic functions of bounded intervals

A bounded interval \( I \) in \( \mathbb{R} \) is a subset of the form \([a, b], (a, b], [a, b) \) or \((a, b)\) for \( a \leq b \), and the length of \( I \), \( l(I) \), is \( b - a \).

The characteristic function of \( I \), \( \chi_I \), is the function

\[
\chi_I : x \mapsto \begin{cases} 
1 & \text{if } x \in I, \\
0 & \text{otherwise}.
\end{cases}
\]

We defined

\[
\int \chi_I = l(I) = b - a.
\]

(b) Step functions

A step function is a linear combination of characteristic functions of bounded intervals; for example,

\[
\phi = c_1 \chi_{I_1} + c_2 \chi_{I_2} + \cdots + c_n \chi_{I_n}.
\]

We defined

\[
\int \phi = c_1 l(I_1) + c_2 l(I_2) + \cdots + c_n l(I_n).
\]

(c) \( L^{\text{inc}} \)

A function \( f \) is in \( L^{\text{inc}} \) if it is the limit almost everywhere of an increasing sequence of step functions, \( \{\phi_n\} \), with bounded integrals. We defined

\[
\int f = \lim \left\{ \int \phi_n \right\}.
\]

(d) \( L^1(\mathbb{R}) \)

A function \( f \) is in \( L^1(\mathbb{R}) \) if it is the difference of two functions in \( L^{\text{inc}} \), that is,

\[
f = g - h, \quad \text{with } g, h \in L^{\text{inc}}.
\]

We defined

\[
\int f = \int g - \int h.
\]

We see that the only place where we used the fact that the domain of the functions under consideration was \( \mathbb{R} \) was in Step (a), and there we used it only in the definition of a bounded interval and the length of a bounded interval. So, once we have defined a bounded interval in \( \mathbb{R}^k \), and the 'length' of such an interval, it looks as though we can work straight through these four steps again. For example, the definition of a null set, implicitly needed in Step (c) because of the phrase 'almost everywhere', was given in terms of a covering by bounded intervals with arbitrarily small total length. With a suitable interpretation of 'length', we should be able to carry over this definition to the \( \mathbb{R}^k \) context.
This, then, is the programme for the present unit. In Section 1 we define a bounded interval in $\mathbb{R}^k$ and also its 'length', and then proceed via Steps (a) and (b), as in Unit 3, Step Functions. In Section 2 we repeat the development of Steps (c) and (d), which we discussed in Unit 4, The Lebesgue Integral on $\mathbb{R}$, to complete the definition of $L^1(\mathbb{R}^k)$, the set of Lebesgue integrable functions from $\mathbb{R}^k$ to $\mathbb{R}$. Finally, in Section 3, we derive a criterion for a function to belong to $L^1(\mathbb{R}^k)$, much as in Section 2 of Unit 5.

This may seem a lot of material for one unit, but once the initial change in our concept of a bounded interval has been made, the generalization of the Lebesgue integral on $\mathbb{R}$ to the Lebesgue integral on $\mathbb{R}^k$ requires no essentially new ideas: the statements and proofs of the new theorems are analogues of those you have already seen in earlier units.

1 Step functions on $\mathbb{R}^k$

1.1 Functions on $\mathbb{R}^k$

For those who are not very confident about dealing with functions of more than one variable, we devote this section to some practice in the manipulation of functions on $\mathbb{R}^k$.

Now

$\mathbb{R}^k = \{(x_1, x_2, \ldots, x_k) : x_1, x_2, \ldots, x_k \in \mathbb{R}\}$;

that is, a point $x \in \mathbb{R}^k$ is an ordered $k$-tuple of real numbers,

$x = (x_1, x_2, \ldots, x_k)$.

A function $f : \mathbb{R}^k \to \mathbb{R}$ associates with each $x \in \mathbb{R}^k$ some real number $f(x) \in \mathbb{R}$. We often write this as

$f(x) = f(x_1, x_2, \ldots, x_k)$.

For example, the functions $f, g$ given by

$f(x_1, x_2) = \begin{cases} 2 & \text{if } x_1 \geq 0 \text{ and } x_2 \geq 0, \\ 0 & \text{otherwise,} \end{cases}$

and

$g(x_1, x_2) = x_1^2 + 2x_1x_2 - 4x_2 \quad (x_1, x_2 \in \mathbb{R})$,

are examples of functions from $\mathbb{R}^2$ to $\mathbb{R}$.

In precisely the same way as for functions with domain $\mathbb{R}$, we can define the functions

$|f_1|, \quad f_1^+, \quad f_1^-, \quad \max\{f_1, f_2\}, \quad \min\{f_1, f_2\}$

for functions $f_1, f_2$ with domain $\mathbb{R}^k$; and, in particular, we can define $f_1 \geq f_2$ if and only if $f_1(x) \geq f_2(x)$ for all $x \in \mathbb{R}^k$.

We can add such functions and multiply them by real numbers, so the set of all such functions from $\mathbb{R}^k$ to $\mathbb{R}$ forms a vector space over the field $\mathbb{R}$.

For example, with the functions $f$ and $g$ defined above:

(a) $(f + g)(x_1, x_2) = \begin{cases} 2 + x_1^2 + 2x_1x_2 - 4x_2 & \text{if } x_1 \geq 0 \text{ and } x_2 \geq 0, \\ x_1^2 + 2x_1x_2 - 4x_2 & \text{otherwise}; \end{cases}$

(b) $|f| = f$ and $f^+ = f$;

(c) $f^-$ is the zero function;

(d) $f \geq 0$, that is, $f(x) \geq 0$ for all $x \in \mathbb{R}^2$. 

5
**Exercise 1**

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f: (x_1, x_2) \mapsto \begin{cases} x_1 & \text{if } x_1 \geq 0, \\ 1 & \text{otherwise}, \end{cases}$$

and $g: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$g: (x_1, x_2) \mapsto \begin{cases} x_2 & \text{if } x_2 \geq 0, \\ 1 & \text{otherwise}. \end{cases}$$

(a) Describe $2f - g$.

(b) Which of the functions $f$, $g$, $2f - g$ are greater than the zero function, $0$?

**Solution**

(a) The function $2f - g: \mathbb{R}^2 \to \mathbb{R}$ is given by

$$(x_1, x_2) \mapsto \begin{cases} 2x_1 - x_2 & \text{if } x_1 \geq 0 \text{ and } x_2 \geq 0, \\ 2x_1 - 1 & \text{if } x_1 \geq 0 \text{ and } x_2 < 0, \\ 2 - x_2 & \text{if } x_1 < 0 \text{ and } x_2 \geq 0, \\ 1 & \text{otherwise}. \end{cases}$$

(b) From their definitions, we see that $f \geq 0$ and $g \geq 0$, but we cannot say that $2f - g$ is greater than zero. In fact, since $(x_1, 0)$ is mapped to $2x_1$ for $x_1 \geq 0$ and $(0, x_2)$ is mapped to $-x_2$ for $x_2 \geq 0$, the function $2f - g$ is not bounded above or below.

**Exercise 2**

Let $f: \mathbb{R}^2 \to \mathbb{R}$, $g: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f: (x_1, x_2) \mapsto x_1 + x_2,$$

$$g: (x_1, x_2) \mapsto 1.$$ 

Describe $|f|$, $f^+$, $f^-$, $\max\{f, g\}$ and $\min\{f, g\}$.

**Solution** The function $|f|: \mathbb{R}^2 \to \mathbb{R}$ is given by

$$|f|: (x_1, x_2) \mapsto \begin{cases} x_1 + x_2 & \text{if } x_1 + x_2 \geq 0, \\ -x_1 - x_2 & \text{otherwise}. \end{cases}$$

![Figure 1](image)
The function $f^+: \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f^+ : (x_1, x_2) \mapsto \begin{cases} 
    x_1 + x_2 & \text{if } x_1 + x_2 \geq 0, \\
    0 & \text{otherwise.}
\end{cases}$$

Figure 2

The function $f^-: \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f^- : (x_1, x_2) \mapsto \begin{cases} 
    0 & \text{if } x_1 + x_2 \geq 0, \\
    -x_1 - x_2 & \text{otherwise.}
\end{cases}$$

Figure 3
The function \( \max\{f, g\} : \mathbb{R}^2 \to \mathbb{R} \) is given by
\[
\max\{f, g\} : (x_1, x_2) \mapsto \begin{cases} 
  x_1 + x_2 & \text{if } x_1 + x_2 \geq 1, \\
  1 & \text{otherwise.}
\end{cases}
\]

Figure 4

The function \( \min\{f, g\} : \mathbb{R}^2 \to \mathbb{R} \) is given by
\[
\min\{f, g\} : (x_1, x_2) \mapsto \begin{cases} 
  1 & \text{if } x_1 + x_2 \geq 1, \\
  x_1 + x_2 & \text{otherwise.}
\end{cases}
\]

Figure 5

1.2 \textbf{Intervals and step functions in } \mathbb{R}^k

In this section we cover the first step in the construction of the Lebesgue integral on \( \mathbb{R}^k \), namely Step (a) dealing with characteristic functions of bounded intervals.
Notes

1 Page 70, line -10. In M203 you met the scalar product of two vectors in \( \mathbb{R}^3 \). This can be generalized to the Euclidean inner product on \( \mathbb{R}^k \), which is the function 
\[
\mathbb{R}^k \times \mathbb{R}^k \longrightarrow \mathbb{R},
\]
\[
(x_1,x_2,\ldots,x_k) \cdot (y_1,y_2,\ldots,y_k) = x_1y_1 + x_2y_2 + \cdots + x_ky_k.
\]
The inner product can be used to define a norm, known also as the Euclidean norm,
\[
\mathbb{R}^k \longrightarrow \mathbb{R},
\]
\[
x \mapsto \sqrt{x \cdot x} = ||x||.
\]
From this we can produce a metric or distance function,
\[
d : \mathbb{R}^k \times \mathbb{R}^k \longrightarrow \mathbb{R},
\]
\[
d : (x,y) \mapsto ||x - y||,
\]
which satisfies all the usual properties that we expect of a distance function, including the triangle inequality. Here Weir is using \( || \) rather than \( || \).

2 Page 71, line 15. By a face (or edge) of a bounded interval \( I \), where \( I \) is defined by the inequalities
\[
\{ x : a_i < x_i < b_i, i = 1, \ldots, k \},
\]
we mean an interval obtained by replacing one or more of the inequalities by equalities of the form
\[
a_i \leq x_i \leq a_i \quad \text{i.e. } a_i = x_i \quad \text{or} \quad b_i \leq x_i \leq b_i \quad \text{i.e. } b_i = x_i.
\]
See Examples 1–3.

3 Page 72, line 20. The set of all step functions on \( \mathbb{R}^k \) is defined to be the set of all finite linear combinations of characteristic functions of bounded intervals in \( \mathbb{R}^k \) and so is the vector space spanned by this set of characteristic functions.

4 Page 72, the proof of Proposition 4.1.1. The proof of this proposition mirrors the proof of Proposition 3.1.1 (Weir: page 25).

We consider Proposition 4.1.1 for a step function on \( \mathbb{R}^2 \) in Example 4.

Example 1 Let \( k = 1 \), and \( I = [a,b] \).

![Figure 6](image)

The only two non-trivial faces are
\[
I' = \{ x : a \leq x \leq a \} = \{ a \} \quad \text{and} \quad I'' = \{ x : b \leq x \leq b \} = \{ b \},
\]
the ends of the interval.
We have
\[
m(I') = (a - a) = 0,
\]
\[
m(I'') = (b - b) = 0. \quad \Box
\]
Example 2 For $k = 2$ and $I = \{(x_1, x_2): a_1 \leq x_1 < b_1, a_2 \leq x_2 \leq b_2\}$, typical faces are of the form

\[
I' = \{(x_1, x_2): x_1 = a_1, a_2 \leq x_2 \leq b_2\},
\]

\[
I'' = \{(x_1, x_2): a_1 \leq x_1 < b_1, x_2 = b_2\},
\]

\[
I''' = \{(x_1, x_2): x_1 = b_1, x_2 = a_2\}.
\]

Figure 7

The intervals $I'$, $I''$ are one-dimensional faces and the interval $I'''$ is a zero-dimensional face, that is, a point.

We have

\[
m(I') = (a_1 - a_1)(b_2 - a_2) = 0,
\]

\[
m(I'') = (b_1 - a_1)(b_2 - b_2) = 0,
\]

\[
m(I''') = (b_1 - b_1)(a_2 - a_2) = 0.
\]

Example 3 For $k = 3$ and $I = \{(x_1, x_2, x_3): a_i \leq x_i \leq b_i, i = 1, 2, 3\}$, we have a much greater number and variety of faces.

For example,

\[
I' = \{x : x \in I, x_3 = b_3\},
\]

\[
I'' = \{x : x \in I, x_1 = b_1, x_3 = a_3\},
\]

\[
I''' = \{x : x \in I, x_1 = a_1, x_2 = a_2, x_3 = a_3\},
\]

are two-, one- and zero-dimensional faces respectively.

Figure 8
We have
\[ m(I') = (b_1 - a_1)(b_2 - a_2)(b_3 - b_3) = 0, \]
\[ m(I'') = (b_1 - b_1)(b_2 - a_2)(a_3 - a_3) = 0, \]
\[ m(I''') = (a_1 - a_1)(a_2 - a_2)(a_3 - a_3) = 0. \]

In this example there are twenty-six faces in all: eight zero-dimensional, twelve one-dimensional and six two-dimensional faces.

You can see that an interval in \( \mathbb{R}^k \) has a vast number of faces for large \( k \), but that each of these faces has zero measure in \( \mathbb{R}^k \).

**Example 4** It may be helpful to see how Proposition 4.1.1 works for a step function on \( \mathbb{R}^2 \).

Let
\[ I_1 = \{(x_1, x_2) \in \mathbb{R}^2 : 1 < x_1 < 3, 1 \leq x_2 \leq 3\}, \]
\[ I_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 2 < x_1 < 3, 1 \leq x_2 < 4\}, \]
and take
\[ \phi = 3\chi_{I_1} - 2\chi_{I_2} \]
as a step function on \( \mathbb{R}^2 \).

We can visualize \( I_1 \) and \( I_2 \) as rectangles in the plane:

![Figure 9](image)

Here the dashed edges are not included in the intervals. The bounding hyperplanes are the lines \( x_1 = 1, x_1 = 2, x_1 = 3 \) and \( x_2 = 1, x_2 = 3, x_2 = 4 \), so that the plane is dissected into forty-nine disjoint intervals of which twenty-five are bounded:

![Figure 10](image)
The following are typical examples of these bounded intervals:

\[I' = \{(x_1, x_2) \in \mathbb{R}^2 : 1 < x_1 < 2 \text{ and } 3 < x_2 < 4\},\]

\[I'' = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 3 \text{ and } 1 < x_2 < 3\},\]

\[I''' = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 2 \text{ and } x_2 = 3\}.

There are nine ‘point’ bounded intervals like \(I'''\), twelve ‘line’ bounded intervals like \(I''\), and four ‘box’ bounded intervals like \(I'\).

However, of these twenty-five bounded intervals in \(\mathbb{R}^2\), our step function takes non-zero values on only ten of these intervals. If we put

\[J_1 = \{(2, 3)\}, \quad J_5 = \{(x_1, 3) : 2 < x_1 < 3\},\]

\[J_2 = \{(2, 1)\}, \quad J_6 = \{(2, x_2) : 1 < x_2 < 3\},\]

\[J_3 = \{(x_1, 1) : 1 < x_1 < 2\}, \quad J_7 = \{(x_1, x_2) \in \mathbb{R}^2 : 1 < x_1 < 2 \text{ and } 1 < x_2 < 3\},\]

\[J_4 = \{(x_1, 1) : 2 < x_1 < 3\}, \quad J_8 = \{(x_1, x_2) \in \mathbb{R}^2 : 2 < x_1 < 3 \text{ and } 1 < x_2 < 3\},\]

\[J_5 = \{(x_1, 3) : 1 < x_1 < 2\}, \quad J_9 = \{(x_1, x_2) \in \mathbb{R}^2 : 2 < x_1 < 3 \text{ and } 3 < x_2 < 4\},\]

\[J_{10} = \{(x_1, x_2) \in \mathbb{R}^2 : 2 < x_1 < 3 \text{ and } 3 < x_2 < 4\},\]

then

\[\chi_{I_1} = \chi_{J_1} + \chi_{J_2} + \chi_{J_3} + \chi_{J_4} + \chi_{J_5} + \chi_{J_6} + \chi_{J_7} + \chi_{J_8} + \chi_{J_9},\]

and

\[\chi_{I_2} = \chi_{J_2} + \chi_{J_3} + \chi_{J_4} + \chi_{J_5} + \chi_{J_6} + \chi_{J_7} + \chi_{J_8} + \chi_{J_9} + \chi_{J_{10}},\]

so

\[\phi = 3\chi_{J_1} + 3\chi_{J_2} + 3\chi_{J_3} + 3\chi_{J_4} + 3\chi_{J_5} + 3\chi_{J_6} + 3\chi_{J_7} + 3\chi_{J_8} + \chi_{J_9} - 2\chi_{J_{10}}.\]

This is the representation of \(\phi\) as a linear combination of characteristic functions of disjoint intervals that we obtain by following the proof of Proposition 4.1.1, but of course there are many other representations. For example, if

\[K = \{(x_1, x_2) \in \mathbb{R}^2 : 2 < x_1 < 3, 1 \leq x_2 \leq 3\}\]

then

\[\chi_K = \chi_{J_1} + \chi_{J_2} + \chi_{J_3},\]

and so

\[\phi = 3\chi_{J_1} + 3\chi_{J_2} + 3\chi_{J_3} + 3\chi_{J_4} + 3\chi_{J_5} + 3\chi_{J_6} + 3\chi_{J_7} + 3\chi_{J_8} + \chi_{J_9} - 2\chi_{J_{10}}\]

is another such representation for \(\phi\). \qed

**Exercise 3**

Weir: page 76, Exercise 1, the first part only. (Here Weir represents a point in \(\mathbb{R}^2\) by \((x, y)\) instead of \((x_1, x_2)\).)

**Solution** This type of question can quickly become tedious unless it is approached systematically. First draw a diagram of the intervals given in the question; for the first part we need only \(I_1\) and \(I_2\).

![Figure 11](image-url)
If you are in any doubt as to how to proceed, follow the steps in the proof of Proposition 4.1.1. However, the diagram suggests that we use the following intervals:

\[ J_1 = \{(x, y) : 1 \leq x < 2 \text{ and } 0 < y \leq 1\}, \]
\[ J_2 = \{(x, y) : 1 < x < 2 \text{ and } 1 < y < 2\}, \]
\[ J_3 = \{(x, y) : 1 \leq x < 2 \text{ and } 2 \leq y \leq 3\}, \]
\[ J_4 = \{(1, y) : 1 < y < 2\}, \]
\[ J_5 = \{(x, y) : 2 \leq x < 3 \text{ and } 1 < y < 2\}. \]

The above intervals are disjoint, and

\[ x_{J_1} = x_{J_2} + x_{J_3} + x_{J_4} + x_{J_5}, \]
\[ x_{J_2} = x_{J_2} + x_{J_4}, \]

so

\[ x_{J_1} - x_{J_2} = x_{J_1} + x_{J_3} + x_{J_4} - x_{J_5}. \]

### 1.3 The integral for step functions on \(\mathbb{R}^k\)

In this section, we implement Step (b) in our strategy by defining the integral of a step function.

**Read** Weir: Chapter 4, page 73, line 3, to page 75, end of Subsection 4.1.

**Notes**

1. **Page 74, Figure 28.** The set \(J\) is the line-segment dividing the rectangle \(I\) into two regions, \(I'\) and \(I''\).

2. **Page 74, line 5.** Since \(I = I' \cup J \cup I''\) and the intervals \(I', J, I''\) are disjoint,

\[ x_I = x_{I'} + x_J + x_{I''}. \]

So for the definition of the integral to be consistent we need

\[ \int x_I = \int x_{I'} + \int x_J + \int x_{I''}; \]

that is, \(m(I) = m(I') + m(I'')\), since \(m(J) = 0\).

3. **Page 74, Proposition 4.1.2.** This is the analogue of Proposition 3.1.2 (**Weir: page 27**). It enables us to show that the definition of the integral of a step function is independent of the representation of the step function as a linear combination of characteristic functions of disjoint bounded intervals. Once we have established this consistency, it is then immediate from our definition that the integral is a linear operator on the vector space of step functions.

**Exercise 4**

Extend the notation and Exercises 12 and 13 of Section 1 of Unit 3 to the case of functions from \(\mathbb{R}^k\) to \(\mathbb{R}\). (This is **Weir: page 76, Exercises 3.**)

**Solution** Solution 12 of Unit 3 requires only trivial changes. In Solution 13, we now require Proposition 4.1.1 instead of Proposition 3.1.1, and \(m(I_i)\) instead of \(I(I_i)\).
Exercise 5

Extend the notation and Exercises 14–19 of Section 1 of Unit 3 to the collection $\mathcal{R}$ of elementary figures in $\mathbb{R}^k$. (This is Weir: page 76, Exercise 4.)

Solution In Exercise 14, $A$ is now a subset of $\mathbb{R}^k$, and $\mathcal{R}$ now denotes the set of elementary figures in $\mathbb{R}^k$. In Exercises 14–19 and their solutions, we now require Propositions 4.1.1 and 4.1.3 in place of Propositions 3.1.1 and 3.1.3 respectively, and $m(I_i)$ instead of $l(I_i)$.

Exercise 6

Weir: page 76, Exercise 5.

Solution Suppose that $\phi = \sum_{i=1}^n c_i x_{I_i}$, then

$$\psi = \sum_{i=1}^n c_i x_{J_i},$$

where $J_i = \{x \in \mathbb{R}^k : x + a \in I_i\}$. In other words, if $a = (a_1, a_2, \ldots, a_k)$ and $I_i$ is specified by the inequalities

$$b_i < x_i < d_i, \quad i = 1, 2, \ldots, k,$$

then $J_i$ is specified by the inequalities

$$b_i - a_i < x_i < d_i - a_i, \quad i = 1, 2, \ldots, k.$$

In such a case,

$$m(J_i) = (d_1 - a_1 - b_1 + a_1)(d_2 - a_2 - b_2 + a_2) \cdots (d_k - a_k - b_k + a_k)$$

so

$$\int \psi = \sum_{i=1}^n c_i m(J_i) = \sum_{i=1}^n c_i m(I_i) = \int \phi.$$

Similarly, if $\theta$ is given by $\theta(x) = \phi(rx)$, then

$$\theta = \sum_{i=1}^n c_i x_{K_i},$$

where $K_i = \{x : rx \in I_i\}$. Now if $r > 0$, then $K_i$ is specified by the inequalities

$$b_i/r < x_i < d_i/r, \quad i = 1, 2, \ldots, k,$$

and if $r < 0$, $K_i$ is specified by the inequalities

$$-d_i/|r| < x_i < -b_i/|r|, \quad i = 1, 2, \ldots, k.$$

In either case,

$$m(K_i) = (1/|r|^k)(d_1 - b_1)(d_2 - b_2) \cdots (d_k - b_k)$$

so

$$\int \theta = \sum_{i=1}^n c_i m(K_i)$$

$$= \int \phi.$$
The integral on $\mathbb{R}^k$

We now turn to the analogue of Unit 4, *The Lebesgue Integral on $\mathbb{R}$*, and extend the definition of the Lebesgue integral to $L^{inc}(\mathbb{R}^k)$ and $L^1(\mathbb{R}^k)$. Again, we have seen all the steps before.

We begin by investigating the possibility of approximating functions by increasing sequences of step functions with bounded integrals. It again turns out that, to obtain as large a class of functions as possible in this way, we must talk about convergence *almost everywhere*. That is, convergence except possibly on null sets, where the concept of a null set on $\mathbb{R}^k$ is analogous to that of a null set on $\mathbb{R}$.

The set of functions we can approximate in this way is called $L^{inc}(\mathbb{R}^k)$; we define the integral of a function $f$ in $L^{inc}$ by the equation

$$\int f = \lim \left\{ \int \phi_n \right\},$$

where $\{\phi_n\}$ is a suitable sequence of step functions approximating $f$.

Finally, we extend the definition of the integral to $L^1(\mathbb{R}^k)$, the smallest vector space containing $L^{inc}(\mathbb{R}^k)$. $L^1(\mathbb{R}^k)$ is the space of Lebesgue integrable functions on $\mathbb{R}^k$.

We developed the integral for functions from $\mathbb{R}$ to $\mathbb{R}$ in such a way as to ensure that the integral had certain nice properties. This section ends by showing that our new integral for functions from $\mathbb{R}^k$ to $\mathbb{R}$ also has these properties.

---

*Read* Weir: Chapter 4, page 77, beginning of Subsection 4.2 to page 79, line 2.

**Notes**


2. Page 78, lines 4 to 13. As in Unit 4, we expect you to appreciate the need for Lemma 1, though you are not expected to be able to reproduce the details.

3. Page 78, line 15. $L^1(\mathbb{R}^k)$ is the vector space spanned by $L^{inc}(\mathbb{R}^k)$; the construction is the same as that of $L^1(\mathbb{R})$ from $L^{inc}(\mathbb{R})$.

4. Page 78, Theorem 4.2.2. The proof is a repeat of that of Theorem 3.2.2 on Weir: page 37.

5. Page 79, Theorem 4.2.3. The proof is a repeat of that of Theorem 3.2.3 on Weir: page 38.

We summarize the definitions and theorems introduced in this section.

**Definition T.6.2(1)** The space $L^{inc}(\mathbb{R}^k)$ consists of those functions $f: \mathbb{R}^k \to \mathbb{R}$ for which there exists a sequence of step functions $\{\phi_n\}$ in $\mathbb{R}^k$ with the properties:

- (a) $\{\phi_n\}$ is increasing, i.e. $\phi_n \leq \phi_{n+1}$ for all $n$;
- (b) $\{\phi_n\}$ converges to $f$ almost everywhere, i.e. $\{\phi_n(x)\}$ converges to $f(x)$ for all $x \in \mathbb{R}^k$, except those $x$ in some null set;
- (c) $\{\int \phi_n\}$ is a convergent sequence of real numbers.
**Definition T.6.2(2)** For \( f \in L^{\infty}(\mathbb{R}^k) \) we define the integral as follows:

(a) first we find a sequence of step functions \( \{ \phi_n \} \) approximating \( f \) in the way described in Definition T.6.2(1);

(b) then we set

\[
\int f = \lim \left\{ \int \phi_n \right\}.
\]

**Definition T.6.2(3)** \( L^1(\mathbb{R}^k) = \{ f: \mathbb{R}^k \to \mathbb{R} : f = g - h, \text{ with } g, h \in L^{\infty}(\mathbb{R}^k) \} \).

**Definition T.6.2(4)** If \( f \in L^1(\mathbb{R}^k) \) and \( f = g - h, \text{ with } g, h \in L^{\infty}(\mathbb{R}^k) \), then we define

\[
\int f = \int g - \int h.
\]

**Theorem T.6.2(1)** \( L^1(\mathbb{R}^k) \) is a vector space.

**Theorem T.6.2(2)** The operator \( f: L^1(\mathbb{R}^k) \to \mathbb{R} \) is well-defined.

---

**Exercise 7**

Show that any countable set in \( \mathbb{R}^k \) is null. (Hint: Generalize the proof of Proposition 2.2.2 on Weir: page 18.)

**Solution** Let \( x_1, x_2, x_3, \ldots \) be the countable set of points in \( \mathbb{R}^k \) and let \( \varepsilon > 0 \) be given; then we cover \( x_i \) by a bounded interval \( I_i \) in \( \mathbb{R}^k \) with \( m(I_i) = \varepsilon/2^i \), mirroring the proof of Proposition 2.2.2 on Weir: page 18. We do this by taking \( I_i \) to be the interval specified by the inequalities

\[ x_{ij} - \frac{1}{2} \frac{\varepsilon}{2^i} < x_{ij} < x_{ij} + \frac{1}{2} \frac{\varepsilon}{2^i}, \quad j = 1, 2, \ldots, k, \]

where \( x_i = (x_{i1}, x_{i2}, \ldots, x_{ik}) \). That is, we are taking \( I_i \) to be a 'box' with centre \( x_i \) and side length \( \frac{\varepsilon}{2^i} \). We have \( m(I_i) = \varepsilon/2^i \) and

\[ \sum m(I_i) = \sum \varepsilon/2^i < \varepsilon, \]

so the countable set is null, as required.

---

**Exercise 8**

*Weir*: page 82, Exercise 1.

**Solution** See *Weir*: page 250.

---

**Exercise 9**

*Weir*: page 82, Exercise 2.

**Solution** See *Weir*: page 250. Here Proposition 4.2.2 is being used three times. First consider the set of lines on which every point \((x,y)\) has \(x\) rational. These are lines parallel to the \(y\)-axis and there is one such line for each point \((x,0)\) with \(x\) rational. That is, there are countably many such lines each of which is null, so, from Proposition 4.2.2, their union is a null set \(A\). Repeating the argument with lines parallel to the \(x\)-axis and using Proposition 4.2.2 again, we see that their union is another null set \(B\). A final application of Proposition 4.2.2 shows that \(A \cup B\) is null.

---

**Exercise 10**

*Weir*: page 82, Exercise 3.
Solution If \( I \) is the interval given by \( 0 < x < 1 \) and \( 0 < y < 1 \), and \( C = I \setminus S \), then \( (x, y) \in C \) if and only if \( x \) or \( y \) is rational. That is, \( C \) is a subset of the set considered in Exercise 9, which was shown to be null, so \( C \) is also a null set in \( \mathbb{R}^2 \). That is, \( \chi_S = \chi_I \) a.e. so, from Theorem 4.2.3 (Weir: page 79), it follows that

\[
\int \chi_S = \int \chi_I = 1.
\]

**Exercise 11**

Show that the points of a straight line in \( \mathbb{R}^2 \) form a null set.

**Solution** Let \( S \) be the set of points of a straight line of slope \( m \). Then \( S \) is a countable union of segments such as \( S_n, S = \bigcup_n S_n \), and, by Proposition 4.2.2, it is sufficient to show that \( S_n \) is a null set.

![Figure 12](image)

Figure 12

We may cover the set \( S_n \) by \( N \) intervals (rectangles) of \( \mathbb{R}^2 \), each of which has the side parallel to the \( x \)-axis of length \( l \), where \( Nl = 1 \).

Given any \( \varepsilon > 0 \), if we choose \( l < \varepsilon/m \), then the measure of the union of this set of rectangles is less than \( \varepsilon \), hence \( S_n \) is a null set.

### 3 The definite integral

In this section we round off the development of the Lebesgue integral on \( \mathbb{R}^k \) by generalizing some results from Unit 5.

**Read** Weir: Chapter 4, page 79, line 3, to page 81, end of Subsection 4.2.

**Notes**

1. Page 79, lines -16 to -6. This definition of continuity will be found in M337, Complex Analysis, for the complex plane. Specialization to the real numbers gives the case \( k = 1 \). The extension to general \( k \) does not introduce any additional complications.

**Exercise 12**

Solution We illustrate the case in which \( r > 1 \):

![Diagram 1](image1)

![Diagram 2](image2)

**Figure 13**
The sequence of step functions \( \{\chi_{S_n}\} \) is increasing and, if \( t \) is any integer greater than \( r \), we have

\[
\int \chi_{S_n} \leq 4t^2.
\]

So \( \{\chi_{S_n}\} \) converges almost everywhere.

Now because \( \{\chi_{S_n}\} \) is an increasing sequence, if \( \chi_{S_m}(x) = 1 \) for some \( n \) and some \( x \), then \( \chi_{S_n}(x) = 1 \) for all \( m \geq n \). That is, \( \{\chi_{S_n}(x)\} \) converges to \( 1 \). This means that if \( A = \bigcup_n S_n \) then \( \{\chi_{S_n}\} \) converges to \( \chi_A \). We are asked to show that this sequence converges to \( \chi_S \) and for this we have only to show that \( A = S \). Now each \( S_n \subset S \) from the construction, so

\[ A \subset S. \]

Now let \( x \in S \); if we can show that \( x \in S_n \) for some \( n \), \( x \) will be in \( A \) and so

\[ S \subset A, \]

and hence \( A = S \).

![Diagram 3](image3)

**Figure 15**
Suppose \( x = (x_1, y_1) \), \( |x| < r \), and the 'horizontal' and 'vertical' distances of the point \( x \) from the circle are \( a \) and \( b \) respectively. Then there exists a positive integer \( N \) such that \( |a| > 2/2^N \) and \( |b| > 2/2^N \).

We now show that \( x \in S_N \).
If we take \( n_1, n_2 \) as the least positive integers for which
\[
0 < |x_1| < n_1/2^N
\]
and
\[
0 < |y_1| < n_2/2^N,
\]
then \( z \) is contained in the rectangle with centre the origin and side lengths \( 2n_1/2^N \), \( 2n_2/2^N \), as shown in the diagram. This rectangle is composed of squares of side length \( 1/2^N \), and it is therefore contained in \( S_N \); hence \( z \in S_N \).

**Exercise 13**

*Weir: page 82, Exercise 8.* In the first line of this exercise there is a misprint:

for \( \mathbb{R}^2 \) read \( \mathbb{R} \).

**Solution** See *Weir: page 251.*

---

### 4 Summary of the text

In this unit we described how the results of the previous units can be generalized to define an integral for suitable functions defined on \( \mathbb{R}^k \). The set of these Lebesgue integrable functions is called \( L^1(\mathbb{R}^k) \).

The unit followed a familiar pattern. We began with characteristic functions of intervals and their integrals and extended this to step functions and their integrals. We then discussed how we could use sequences of step functions to extend the definition of the integral to as many functions as possible, and we discovered that we should use increasing sequences of step functions with bounded integrals, and that such sequences converge almost everywhere. We thus obtained the set of functions \( L^{inc}(\mathbb{R}^k) \) and the definition of the integral of a function \( f \) in \( L^{inc}(\mathbb{R}^k) \); \( f \) is in \( L^{inc}(\mathbb{R}^k) \) if \( f \) is the limit almost everywhere of a sequence of step functions \( \{\phi_n\} \) of the kind described above, and we define
\[
\int f = \lim \left\{ \int \phi_n \right\}.
\]

The last step was to go from the class of functions \( L^{inc}(\mathbb{R}^k) \) to the smallest vector space \( L^1(\mathbb{R}^k) \) containing it, by considering the set of differences of functions in \( L^{inc}(\mathbb{R}^k) \).

Finally, we introduced \( L^1(I) \), the set of functions integrable on some bounded interval \( I \). We defined \( f \in L^1(I) \) if and only if \( f x_I \in L^1(\mathbb{R}^k) \), and showed that, if \( f \) is a bounded function, continuous almost everywhere on \( I \), then \( f \in L^1(I) \).

In summary, **Unit 6** lays the foundations for Lebesgue integration on \( \mathbb{R}^k \) with \( k > 1 \) and shows how the definition and results obtained for \( L^1(\mathbb{R}) \) extend to higher dimensions. The nitty-gritty of this development is sometimes more tedious than illuminating — the important thing is an understanding of the results.

Some of you may wish to note that although the theory of later units applies to integration on \( \mathbb{R}^k \), the discussion for \( k > 1 \) does not use any special methods beyond those needed for integration on \( \mathbb{R} \). So you can work through these later units with confidence even if your grasp of **Units 6 and 7** is incomplete.

In the following subsections references to the set book or this text are preceded by \( W \) or \( T \) respectively. For example, a reference to page 9 of the set book is indicated by \([W9]\) and to Section 3 of this text by \([T3]\).
4.1 Notation

\[ f \geq g \] [T1]
\[ |f| \] [T1]
\[ f^+ \] [T1]
\[ f^- \] [T1]
\[ \max\{f, g\} \] [T1]
\[ \min\{f, g\} \] [T1]
\[ |x - y| \] [W70]
\[ a_i < k_i < b_i \] [T1]
\[ m(I) \] [W71]
\[ \int \phi \] [W73]
\[ \mathcal{R} \] [W75]
\[ m(S) \] [W75]
a.e. in \( \mathbb{R}^k \) [W77]
\[ L^{\infty}(\mathbb{R}^k) \] [W78]
\[ L^1(\mathbb{R}^k) \] [W78]
\[ \int_I f \] [W78]
\[ \int_I \phi \] [W79]
\[ L^1(I) \] [W79]

4.2 Glossary

almost everywhere in \( \mathbb{R}^k \) [W77]
area [W71]
bounded interval in \( \mathbb{R}^k \) [W70]
bounding hyperplane [W71]
closed interval in \( \mathbb{R}^k \) [W71]
continuity of function on \( \mathbb{R}^k \) [W79]
cube [W71]
elementary figure [W75]
Euclidean norm [W70]
Euclidean space [W70]
integral of function on \( \mathbb{R}^k \) [W78]
integral of function on an interval in \( \mathbb{R}^k \) [W79]
integral of step function on \( \mathbb{R}^k \) [W73]
measure of bounded interval in \( \mathbb{R}^k \) [W71]
null set in \( \mathbb{R}^k \) [W77]
open interval in \( \mathbb{R}^k \) [W71]
point in \( \mathbb{R}^k \) [W70]
step function on \( \mathbb{R}^k \) [W72]
unbounded interval in \( \mathbb{R}^k \) [W71]
volume [W71]

4.3 Results

**Proposition 4.1.1** [W72] Any step function in \( \mathbb{R}^k \) may be expressed as a finite linear combination of characteristic functions of disjoint bounded intervals.

**Proposition 4.1.2** [W74] The definition of the integral of a step function in \( \mathbb{R}^k \) is consistent, that is, if

\[ \phi = c_1 x_{I_1} + c_2 x_{I_2} + \cdots + c_r x_{I_r} \]
\[ = d_1 x_{J_1} + d_2 x_{J_2} + \cdots + d_s x_{J_s}, \]

then

\[ \int \phi = c_1 m(I_1) + c_2 m(I_2) + \cdots + c_r m(I_r) \]
\[ = d_1 m(J_1) + d_2 m(J_2) + \cdots + d_s m(J_s). \]

**Proposition 4.1.3** [W75] If \( \phi \geq \psi \), then \( \int \phi \geq \int \psi \).

**Proposition 4.2.1** [W77] Any countable set in \( \mathbb{R}^k \) is null.

**Proposition 4.2.2** [W77] The union of a sequence of null sets in \( \mathbb{R}^k \) is itself a null set.

**Theorem 4.2.1** [W77] If \( \{\phi_n\} \) is an increasing sequence of step functions on \( \mathbb{R}^k \) for which the sequence \( \{\int \phi_n\} \) converges, then \( \{\phi_n\} \) converges almost everywhere (i.e. \( \{\phi_n(x)\} \) converges for almost all \( x \)).

**Theorem 4.2.1 (Converse)** [W77] If \( S \) is a null set in \( \mathbb{R}^k \), then there is an increasing sequence \( \{\psi_n\} \) of step functions for which the sequence \( \{\int \psi_n\} \) converges and such that \( \{\psi_n(x)\} \) diverges for every \( x \) in \( S \).
Theorem 4.2.2 [W78]  
(a) If \( f_1, f_2 \in L^1 \) and \( c_1, c_2 \in \mathbb{R} \), then \( c_1 f_1 + c_2 f_2 \in L^1 \) and
\[
\int (c_1 f_1 + c_2 f_2) = c_1 \int f_1 + c_2 \int f_2.
\]
(b) If \( f \in L^1 \) and \( f \geq 0 \) almost everywhere, then
\[
\int f \geq 0.
\]
(c) If \( f \in L^1 \), then \( |f| \in L^1 \) and
\[
\left\| f \right\| \leq \int |f|.
\]

**Theorem T.6.2(1) [T2]**  \( L^1(\mathbb{R}^k) \) is a vector space.

**Theorem T.6.2(2) [T2]** The operator \( f : L^1(\mathbb{R}^k) \to \mathbb{R} \) is well-defined.

**Theorem 4.2.3 [W79]** If \( f_1 \in L^1 \) and \( f_2 = f_1 \) almost everywhere, then \( f_2 \in L^1 \) and
\[
\int f_2 = \int f_1.
\]

**Theorem 4.2.4 [W81]** Let \( f : \mathbb{R}^k \to \mathbb{R} \) be a function which vanishes outside a bounded interval \( I \) of \( \mathbb{R}^k \). If \( f \) is bounded and if the points of discontinuity of \( f \) form a null set in \( \mathbb{R}^k \), then \( f \in L^1(\mathbb{R}^k) \).

**Theorem 4.2.4' [W81]** Let \( I \) be a bounded interval in \( \mathbb{R}^k \). If \( f \) is bounded on \( I \) and if the points of discontinuity of \( f \) on \( I \) form a null set, then \( f \in L^1(I) \).

**Corollary to Theorem 4.2.4' [W81]** If \( f \) is continuous on the bounded closed interval \( I \), then \( f \) is integrable on \( I \).

## 5 Self-Assessment Questions

### 5.1 Step functions on \( \mathbb{R}^k \)

**SAQ 1**

Let
\[
S = \{(x, y) \in \mathbb{R}^2 : 1 < x \leq 2, 2 \leq y < 3\}
\]
and let \( \phi = \chi_S \). If \( \theta, \psi : \mathbb{R}^2 \to \mathbb{R} \) are defined by
\[
\theta(x, y) = \phi ((x, y) + (1, 1)),
\]
\[
\psi(x, y) = \phi (-2(x, y)),
\]
show that \( \theta \) and \( \psi \) are characteristic functions of bounded intervals in \( \mathbb{R}^2 \).
SAQ 2
Let \( \{S_n\} \) be the following sequence of sets in the plane:

\[ S_1 \]
\[ \frac{1}{2} \quad \frac{1}{2} \]

\[ S_2 \]
\[ 0 \quad \frac{1}{4} \quad 1 \quad 1 \frac{1}{4} \quad 2 \]

\[ S_3 \]
\[ 0 \quad \frac{1}{8} \quad 1 \quad 1 \frac{1}{8} \quad 2 \quad 2 \frac{1}{8} \]

Figure 16
Figure 17
Figure 18

The set \( S_n \) consists of \( n \) squares each of side length \( 1/2^n \) placed to the right of the integer points on the \( x \)-axis. More precisely,

\[ S_n = \{(x, y) : x \in [i, i + 1/2^n], i = 0, 1, \ldots, n - 1, \text{ and } y \in [0, 1/2^n]\}. \]

Now put

\[ \phi_n = x_{S_1} + x_{S_2} + \cdots + x_{S_n}. \]

Show that \( \{\phi_n\} \) is a sequence of step functions in \( \mathbb{R}^2 \) with strictly increasing but bounded integrals that does not converge on the null set

\[ S = \{(i, j) \in \mathbb{R}^2 : j = 0, i = 0, 1, 2, \ldots \}. \]

5.2 The integral on \( \mathbb{R}^k \)

SAQ 3
Let \( f \in L^1(\mathbb{R}^k) \). We proved in Theorem 4.2.2(iii) (Weir: page 78) that \(|f|\) is also in \( L^1(\mathbb{R}^k) \). Use this result to show that \( f^+ \) and \( f^- \) are in \( L^1(\mathbb{R}^k) \), and that if \( g \) belongs to \( L^1(\mathbb{R}^k) \) then so do \( \max\{f, g\} \) and \( \min\{f, g\} \).
SAQ 4
Let $T$ be the triangular region in $\mathbb{R}^2$ specified by

$$T = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y < 1\}.$$ 

Show that $\chi_T$ is in $L^1(\mathbb{R}^2)$ and evaluate $\int \chi_T$.

SAQ 5
Work Weir: page 42, Exercise 5 with $L^1$ replaced by $L^1(\mathbb{R}^k)$.

SAQ 6
Let $f \in L^1(\mathbb{R}^k)$ and $I$ be some bounded interval in $\mathbb{R}^k$. Show that $f \chi_I \in L^1(\mathbb{R}^k)$.

SAQ 7
Let $I, J$ be bounded intervals in $\mathbb{R}^k$ such that $I \subset J$. Use the result of the previous question to show that

$$L^1(J) \subset L^1(I).$$

SAQ 8
Weir: page 82, Exercise 5.

Solutions to Self-Assessment Questions

Solution 1
We have

$$\phi(x, y) = \begin{cases} 1 & \text{if } 1 < x \leq 2 \text{ and } 2 \leq y < 3, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\theta(x, y) = \phi((x, y) + (1, 1)) = \phi(x + 1, y + 1),$$

so

$$\theta(x, y) = \begin{cases} 1 & \text{if } 1 < x + 1 \leq 2 \text{ and } 2 \leq y + 1 < 3, \\ 0 & \text{otherwise}. \end{cases}$$

That is, if

$$U = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1 \text{ and } 1 \leq y < 2\},$$

then $\theta = \chi_U$, and so $\theta$ is a characteristic function of a bounded interval.

Similarly, $\psi = \chi_V$, where

$$V = \{(x, y) \in \mathbb{R}^2 : -1 \leq x < -\frac{1}{2} \text{ and } -\frac{3}{2} < y \leq -1\},$$

so $\psi$ is a characteristic function of a bounded interval.

Solution 2
We have

$$\phi_n = \chi_{S_1} + \chi_{S_2} + \cdots + \chi_{S_n},$$

so $\phi_n$ is a step function, and

$$\int \phi_n = \int \chi_{S_1} + \int \chi_{S_2} + \cdots + \int \chi_{S_n}$$

$$= 1/4 + 2(1/2^2 \times 1/2^2) + 3(1/2^3 \times 1/2^3) + \cdots + n(1/2^n \times 1/2^n)$$

$$= 1/2^2 + 2/2^4 + 3/2^6 + \cdots + n/2^{2n}. $$
Now \(\{\phi_n\}\) is an increasing sequence of step functions because we can think of the construction of \(\phi_n\) as the stacking of 'blocks': at the \(n\)th stage we add \(n\) blocks, one to each of the 'towers' at \(0, 1, 2, \ldots, n - 1\).

We have
\[
\int \phi_{n+1} > \int \phi_n,
\]
because
\[
\int \phi_{n+1} = \int \phi_n + (n + 1)/2^{2(n+1)}.
\]
On the other hand,
\[
\int \phi_n = 1/2^2 + 2/2^4 + \cdots + n/2^{2n} \\
\leq 1/2^2 + 1/2^3 + \cdots + 1/2^n \\
\leq 1/2,
\]
so \(\{\phi_n\}\) is an increasing sequence of step functions with a strictly increasing sequence of bounded integrals.

Finally, consider \(\phi_n(m, 0)\), where \(m\) is a non-negative integer:
\[
\phi_n(m, 0) = \begin{cases} 
0 & \text{if } n < m, \\
0 - m & \text{if } n \geq m,
\end{cases}
\]
so \(\{\phi_n\}\) does not converge on the null set \(S\), as required.

**Solution 3**

These results all follow from the identities
\[
f^+ = \frac{1}{2}(f + |f|),
\]
\[
f^- = \frac{1}{2}(|f| - f),
\]
\[
\max\{f, g\} = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|,
\]
\[
\min\{f, g\} = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|,
\]
which hold for functions from \(\mathbb{R}^k\) to \(\mathbb{R}\).

**Solution 4**

We can follow a similar procedure to that in the solution to Exercise 4.2.6 (Weir: page 82). We consider a sequence of subsets of \(T\), \(\{T_n\}\), where \(T_n\) is the union of squares of side length \(1/2^n\) which fit inside \(T\). For example,
At each stage, we include the left-hand and bottom edges of the squares and exclude the right-hand and top edges.

Now let \( \phi_n = \chi_{T_n} \); then the construction shows that \( \{\phi_n\} \) is an increasing sequence of step functions which converges to \( \chi_T \), because if \((x, y)\) is any point in \( T \), then the perpendicular distance from \((x, y)\) to the line \( x + y = 1 \) is some number \( \delta \) and, taking \( N \) such that
\[
\delta > 1/(2^N \sqrt{2})
\]
we have \((x, y) \in T_N\).

Also
\[
\int \phi_1 = 1/2^2, \quad \int \phi_2 = 1/2^2 + 2/2^4,
\]
and, in general,
\[
\int \phi_n = \int \phi_{n-1} + 2^{n-1}/(2^n \times 2^n)
= \int \phi_{n-1} + 1/2^{n+1}.
\]
That is,
\[
\int \phi_n = 1/2^2 + 1/2^3 + \cdots + 1/2^{n+1} \leq \frac{1}{2},
\]
and so the sequence \( \{\int \phi_n\} \) is bounded and we have produced an increasing sequence of step functions with bounded integrals converging to \( \chi_T \). Thus \( \chi_T \) is integrable and
\[
\int \chi_T = \lim \left\{ \int \phi_n \right\}.
\]
Now,
\[
\lim \left\{ \int \phi_n \right\} = 1/2^2 + 1/2^3 + \cdots + 1/2^n + \cdots = \frac{1}{2},
\]
so
\[
\int \chi_T = \frac{1}{2}.
\]

**Solution 5**
The solution on Weir: page 245 carries over word for word if we interpret \( L^1(\mathbb{R}^k) \) as \( L^1(\mathbb{R}^k) \).
Solution 6

We prove this result using the four steps in the construction of the Lebesgue integral.

(a) Let \( J \) be some bounded interval in \( \mathbb{R}^k \); then \( \chi_J \in L^1(\mathbb{R}^k) \) and

\[
\chi_J \chi_I = \chi_{J \cap I}.
\]

Now \( J \cap I \) is again a bounded interval, because if \( J \) is specified by the inequalities

\[
a_i < x_i < b_i, \quad i = 1, 2, \ldots, k,
\]

and \( I \) is specified by the inequalities

\[
c_i < x_i < d_i, \quad i = 1, 2, \ldots, k,
\]

then the \( x_i \) coordinate of any point in \( J \cap I \) must be in the intersection of the two intervals in \( \mathbb{R} \) specified by \( a_i, b_i \) and \( c_i, d_i \). For example, if

\[
a_i \leq x_i \leq b_i \quad \text{and} \quad c_i \leq x_i \leq d_i,
\]

then the \( x_i \) coordinate of the intersection will be in

\[\left[a_i, b_i\right] \cap \left[c_i, d_i\right],\]

which is an interval in \( \mathbb{R} \).

That is, \( \chi_{J \cap I} \in L^1(\mathbb{R}^k) \) and therefore \( \chi_J \chi_I \in L^1(\mathbb{R}^k) \).

(b) Now let \( \phi \in L^1(\mathbb{R}^k) \) be a step function; then

\[
\phi = c_1 \chi_{I_1} + c_2 \chi_{I_2} + \cdots + c_n \chi_{I_n}
\]

for bounded intervals \( I_1, I_2, \ldots, I_n \), and

\[
\phi \chi_I = c_1 \chi_{I_1} \chi_I + c_2 \chi_{I_2} \chi_I + \cdots + c_n \chi_{I_n} \chi_I = c_1 \chi_{I_1 \cap I} + c_2 \chi_{I_2 \cap I} + \cdots + c_n \chi_{I_n \cap I}.
\]

From part (a), each \( \chi_{I_n \cap I} \) is the characteristic function of a bounded interval, so \( \phi \chi_I \) is also a step function and belongs to \( L^1(\mathbb{R}^k) \).

(c) Suppose that \( f \in L^{\infty}(\mathbb{R}^k) \); then there is an increasing sequence of step functions with bounded integrals, \( \{ \phi_n \} \), which converges almost everywhere to \( f \). From part (b), we know that \( \{ \phi_n \chi_I \} \) is also a sequence of step functions, and if \( x \in I \) then

\[
(\phi_{n+1} \chi_I)(x) = \phi_{n+1}(x) \leq \phi_n(x) = (\phi_n \chi_I)(x);
\]

whilst if \( x \notin I \) then

\[
(\phi_{n+1} \chi_I)(x) = 0 = (\phi_n \chi_I)(x),
\]

so the sequence \( \{ \phi_n \chi_I \} \) is also increasing.

Further, \( \{ \phi_n \chi_I \} \) converges almost everywhere to \( f \chi_I \), because if \( x \notin I \) then \( (f \chi_I)(x) = 0 \) and each \( \phi_n \chi_I \) converges to \( 0 \); whilst if \( x \in I \) and \( \phi_n(x) \) converges to \( f(x) \), we have

\[
(\phi_n \chi_I)(x) = \phi_n(x) \quad \text{and} \quad (f \chi_I)(x) = f(x),
\]

so \( \{ \phi_n \chi_I \} \) converges to \( (f \chi_I)(x) \).

If \( N \) is the null set on which \( \{ \phi_n \} \) does not converge to \( f \), then \( N \cap I \) is the set of points on which \( \phi_n \chi_I \) does not converge to \( f \chi_I \), and \( N \cap I \), considered as a subset of \( N \), is also null.

The last requirement for \( f \chi_I \) to be in \( L^{\infty}(\mathbb{R}^k) \) is that \( \{ \phi_n \chi_I \} \) is bounded. To show this, consider the sequence \( \{ \phi_n - \phi_1 \} \). This is a sequence of positive step functions, and if

\[
\phi_n - \phi_1 = c_1 \chi_{I_1} + c_2 \chi_{I_2} + \cdots + c_n \chi_{I_n},
\]

26
where the intervals \(I_1, I_2, \ldots, I_r\) are disjoint, then each \(c_i \geq 0\), \(i = 1, \ldots, r\), so, from part (a), we have
\[
(\phi_n - \phi_1)\chi_I = c_1\chi_{I_1 \cap I} + c_2\chi_{I_2 \cap I} + \cdots + c_r\chi_{I_r \cap I};
\]
and, since \(m(\chi_{I \cap I}) \leq m(\chi_I)\) and each \(c_i \geq 0\),
\[
\int (\phi_n - \phi_1)\chi_I = \sum c_i m(\chi_{I \cap I}) \\
\leq \sum c_i m(\chi_{I_i}) \\
= \int (\phi_n - \phi_1).
\]
Thus
\[
\int \phi_n \chi_I \leq \int \phi_n - \int \phi_1 + \int \phi_1 \chi_I.
\]
Since the sequence \(\{\int \phi_n\}\) is bounded, so also is
\[
\left\{ \int \phi_n \chi_I \right\}.
\]
This completes the step to show that \(f\chi_I \in L^{inc}(\mathbb{R}^k)\).

(d) Now let \(f \in L^1(\mathbb{R}^k)\); then \(f = g - h\), with \(g, h \in L^{inc}(\mathbb{R}^k)\). It follows that
\[
f\chi_I = g\chi_I - h\chi_I,
\]
and we have shown in part (c) that \(g\chi_I\) and \(h\chi_I\) are in \(L^{inc}(\mathbb{R}^k)\), so
\[
f\chi_I \in L^1(\mathbb{R}^k),
\]
as required.

**Solution 7**

If \(I \subset J\), then \(\chi_J \chi_I = \chi_I\), so if \(f \in L^1(J)\), then \(f\chi_J \chi_I \in L^1(\mathbb{R}^k)\) and \(f\chi_J \chi_I \in L^1(\mathbb{R}^k)\), from SAQ 6. But \(f\chi_J \chi_I = f\chi_I\), so \(f \in L^1(I)\) and we have shown that
\[
L^1(J) \subset L^1(I).
\]

**Solution 8**

This again requires a proof in four steps, but the first two steps have been completed in the exercise quoted.

We introduce two new functions, \(g, h\), from \(\mathbb{R}^k\) to \(\mathbb{R}\) by putting
\[
g(x) = f(x + a) \quad (x \in \mathbb{R}^k),
\]
\[
h(x) = f(rx) \quad (x \in \mathbb{R}^k).
\]
Suppose that \(f \in L^{inc}(\mathbb{R}^k)\); then there exists an increasing sequence of step functions with bounded integrals, \(\{\phi_n\}\), which converges almost everywhere to \(f\).

If we put
\[
\psi_n(x) = \phi_n(x + a) \quad (x \in \mathbb{R}^k),
\]
\[
\theta_n(x) = \phi_n(rx) \quad (x \in \mathbb{R}^k),
\]
then Exercise 4.1.5 (Weir: page 76) shows that \(\{\psi_n\}\), \(\{\theta_n\}\) are sequences of step functions and
\[
\int \psi_n = \int \phi_n,
\]
\[
\int \theta_n = (1/|r|^k) \int \phi_n.
\]
So, since the sequence of integrals \(\{\int \phi_n\}\) is bounded, each of the sequences \(\{\int \psi_n\}\) and \(\{\int \theta_n\}\) is also bounded. It is also clear that, because \(\{\phi_n\}\) is an increasing sequence,
\[
\psi_{n+1}(x) = \phi_n(x + a) \geq \phi_n(x + a) = \psi_n(x)
\]
and

$$\theta_{n+1}(x) = \phi_{n+1}(rx) \geq \phi_{n}(rx) = \theta_{n}(x),$$

so the sequences \(\{\psi_{n}\}\) and \(\{\theta_{n}\}\) are also increasing.

Finally, these sequences converge almost everywhere to \(g\) and \(h\) respectively; for example, if \(\{\phi_{n}\}\) converges at \(x + a\), then \(\{\phi_{n}(x)\}\) converges to \(f(x + a) = g(x)\).

We have shown, therefore, that \(g\) and \(h\) are in \(L^{\text{inc}}(\mathbb{R}^{k})\) and

\[
\int g = \lim \left\{ \int \psi_{n} \right\} = \lim \left\{ \int \phi_{n} \right\} = \int f,
\]

\[
\int h = \lim \left\{ \int \theta_{n} \right\} = \lim \left\{ \frac{1}{|r|^{k}} \int \phi_{n} \right\} = \left( \frac{1}{|r|^{k}} \right) \int f.
\]

For the last step, let \(f \in L^{1}(\mathbb{R}^{k})\) and let \(f = f_{1} - f_{2}\), where \(f_{1}, f_{2} \in L^{\text{inc}}(\mathbb{R}^{k})\). If we introduce the functions \(g, g_{1}, g_{2}, h, h_{1}, h_{2}\), defined as above (for example,

\[
g(x) = f(x + a) \quad (x \in \mathbb{R}^{k}),
\]

\[
h_{1}(x) = f_{1}(rx) \quad (x \in \mathbb{R}^{k}),
\]

we have \(g = g_{1} - g_{2}\), \(h = h_{1} - h_{2}\) and since \(g_{1}, g_{2}, h_{1}, h_{2} \in L^{\text{inc}}(\mathbb{R}^{k})\), it follows that \(g, h \in L^{1}(\mathbb{R}^{k})\).

Now

\[
\int g = \int g_{1} - \int g_{2}
\]

\[
= \int f_{1} - \int f_{2} = \int f
\]

and

\[
\int h = \int h_{1} - \int h_{2}
\]

\[
= \left( \frac{1}{|r|^{k}} \right) \int f_{1} - \left( \frac{1}{|r|^{k}} \right) \int f_{2}
\]

\[
= \left( \frac{1}{|r|^{k}} \right) \int f,
\]

using the previous part of the question.